Combinatorics

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Counting (labeled) trees

“How many different trees can be formed from \( n \) distinct vertices?”
Cayley’s formula:

There are $n^{n-2}$ trees on $n$ distinct vertices.
Prüfer Code

leaf: vertex of degree 1

removing a leaf from $T$, still a tree

$T_5$:

$u_i$: 2, 4, 5, 6, 3, 1
$v_i$: 4, 3, 1, 3, 1, 7

$T_1 = T$;

for $i = 1$ to $n-1$

$u_i$: smallest leaf in $T_i$;

$(u_i, v_i)$: edge in $T_i$;

$T_{i+1} = \text{delete } u_i \text{ from } T_i$;

Prüfer code:

$(v_1, v_2, \ldots, v_{n-2})$
edges of $T$: $(u_i,v_i)$, $1 \leq i \leq n-1$

$v_{n-1} = n$

$u_i$: smallest leaf in $T_i$

a tree has $\geq 2$ leaves

Only need to recover every $u_i$ from $(v_1, v_1, \ldots, v_{n-2})$.

$u_i$ is the smallest number not in

$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

$T:

\begin{align*}
2 & \quad 4 & \quad 3 & \quad 1 & \quad 5 & \quad 7 & \quad 6 \\
\end{align*}$

$u_i: \quad 2, 4, 5, 6, 3, 1$

$v_i: \quad 4, 3, 1, 3, 1, 7$

$(v_1, v_2, \ldots, v_{n-2})$
\[ u_i \text{ is the smallest number not in } \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \]

\[ \forall \text{ vertex } v \text{ in } T, \]
\[ \# \text{ occurrences of } v \text{ in } u_1, u_2, \ldots, u_{n-1}, v_{n-1} : 1 \]
\[ \# \text{ occurrences of } v \text{ in edges } (u_i, v_i), 1 \leq i \leq n-1: \] \[ \text{deg}_T(v) \]

\[ T : \]

\[ u_i: 2, 4, 5, 6, 3, 1 \]
\[ v_i: 4, 3, 1, 3, 1, 7 \]
\[ (v_1, v_2, \ldots, v_{n-2}) \]
\( u_i \) is the smallest number not in
\[ \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \]

\( \forall \) vertex \( v \) in \( T_i \),

\# occurrences of \( v \) in \( u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1} \): \( 1 \)

\# occurrences of \( v \) in edges \((u_j,v_j), i \leq j \leq n-1\): \( \deg_{T_i}(v) \)

\[ T_3 : \]

\begin{align*}
\text{\# occurrences of } v & \text{ in } (v_i, \ldots, v_{n-2}) \\
\text{\# } \deg_{T_i}(v) - 1
\end{align*}

leaf \( v \) of \( T_i \):

\begin{align*}
\text{in } & \{u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}\} \\
\text{not in } & \{v_i, v_{i+1}, \ldots, v_{n-2}\}
\end{align*}

\( u_i \): smallest leaf in \( T_i \)

\begin{align*}
\text{\( u_i \): } & 2, 4, 5, 6, 3, 1 \\
\text{\( v_i \): } & 4, 3, 1, 3, 1, 7 \\
\text{(\( v_1, v_2, \ldots, v_{n-2} \))}
\end{align*}
$u_i$ is the smallest number not in \( \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \)

$T$: 

\[
\begin{array}{c}
2 \\
\text{3} \\
\text{4} \\
\text{5} \\
\text{6} \\
\text{7}
\end{array}
\]

$u_i$: \{2, 4, 5, 6, 3, 1\}
$v_i$: \{4, 3, 1, 3, 1, 7\}

($v_1, v_2, \ldots, v_{n-2}$)

$T = \text{empty graph}$;
$v_{n-1} = n$;
for $i = 1$ to $n-1$
    $u_i$: smallest number not in
    \( \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \)
    add edge $(u_i, v_i)$ to $T$;
Prüfer code is reversible $\implies$ 1-1

Every $(v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}$ is decodable to a tree $\implies$ onto

$T$:

![Diagram of a tree]

$u_i$: 2, 4, 5, 6, 3, 1

$v_i$: 4, 3, 1, 3, 1, 7

$(v_1, v_2, \ldots, v_{n-2})$
Prüfer code is reversible 1-1
every \((v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}\)
is decodable to a tree onto

Cayley’s formula:
There are \(n^{n-2}\) trees on \(n\) distinct vertices.
Double Counting

# of sequences of adding directed edges to an empty graph to form a rooted tree
\( T_n \) : \# of trees on \( n \) distinct vertices.

\# of sequences of adding directed edges to an empty graph to form a rooted tree

From a tree:
- pick a root;
- pick an order of edges.

\[ T_n n(n - 1)! = n!T_n \]
$T_n$: the number of trees on $n$ distinct vertices.

The number of sequences of adding directed edges to an empty graph to form a rooted tree.

From an empty graph:
- add edges one by one.
From an empty graph:

- add edges one by one

Start from $n$ isolated vertices

Each step joins 2 trees.
From an empty graph: • add edges one by one

Start from $n$ rooted trees.
After adding $k$ edges

$n$-k rooted trees

add an edge

any vertex $\rightarrow$ root of another tree $n \rightarrow n-k-1$
# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph:

- add edges one by one

\[
\prod_{k=0}^{n-2} n(n - k - 1)
\]

\[
= n^{n-1} \prod_{k=1}^{n-1} k
\]

\[
= n^{n-2} n!
\]

Start from \(n\) rooted trees.

After adding \(k\) edges

\(n-k\) rooted trees

add an edge

any vertex \(\rightarrow\) root of another tree

\(n\) \(n-k-1\)
# of sequences of adding directed edges to an empty graph to form a rooted tree

From a tree:

- pick a root;
- pick an order of edges.

\[
T_n n(n - 1)! = n!T_n
\]

From an empty graph:

- add edges one by one

\[
\prod_{k=2}^{n} n(k - 1) = n^{n-2}n!
\]

\[
T_n = n^{n-2}
\]
Graph Laplacian

Graph \( G(V,E) \)

adjacency matrix \( A \)

\[
A(i, j) = \begin{cases} 
1 & \{i, j\} \in E \\
0 & \{i, j\} \notin E
\end{cases}
\]

diagonal matrix \( D \)

\[
D(i, j) = \begin{cases} 
\text{deg}(i) & i = j \\
0 & i \neq j
\end{cases}
\]

graph Laplacian \( L \)

\[
L = D - A
\]

\[
L = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 3 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\]
**Graph Laplacian**

**Graph Laplacian** $L$

$$L(i, j) = \begin{cases} \text{deg}(i) & i = j \\ -1 & i \neq j, \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

**quadratic form:**

$$xLx^T = \sum_i d_i x_i^2 - \sum_{ij \in E} x_i x_j = \frac{1}{2} \sum_{ij \in E} (x_i - x_j)^2$$

**incident matrix** $B: n \times m$

$$i \in V, e \in E$$

$$B(i, e) = \begin{cases} 1 & e = \{i, j\}, i < j \\ -1 & e = \{i, j\}, i > j \\ 0 & \text{otherwise} \end{cases}$$

$L = BB^T$
Kirchhoff’s matrix-tree theorem

$L_{i,i}$ : submatrix of $L$ by removing $i$th row and $i$th column

$t(G)$ : number of spanning trees in $G$
Kirchhoff's matrix-tree theorem

$L_{i,i}$ : submatrix of $L$ by removing $i$th row and $i$th column

$t(G)$ : number of spanning trees in $G$

**Kirchhoff's Matrix-Tree Theorem:**

\[ \forall i, \quad t(G) = \det(L_{i,i}) \]
Kirchhoff’s Matrix-Tree Theorem:
\[ \forall i, \quad t(G) = \det(L_{i,i}) \]

\[ C : (n - 1) \times m \]
incident matrix \( B \) removing \( i \)th row

\[ L = BB^T \]

\[ L_{i,i} = CC^T \quad \det(L_{i,i}) = \det(CC^T) = ? \]
Cauchy-Binet Theorem:
\[
\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})
\]

\(A : n \times m\)

\(B : m \times n\)
Cauchy-Binet Theorem:

\[
\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n], S}) \det(B_{S, [n]})
\]

\[
\det(L_{i,i}) = \det(CC^T)
\]

\[
= \sum_{S \in \binom{[m]}{n-1}} \det(C_{[n-1], S}) \det(C_{S, [n-1]}^T)
\]

\[
= \sum_{S \in \binom{[m]}{n-1}} \det(C_{[n-1], S})^2
\]
\[
\det(L_{i,i}) = \sum_{S \in \binom{\{m\}}{n-1}} \det(C_{[n-1],S})^2
\]

\[j \in [n] \setminus \{i\}, e \in S\]

\[C_{[n-1],S}(j, e) = \begin{cases} 
1 & e = \{j, k\}, j < k \\
-1 & e = \{j, k\}, j > k \\
0 & \text{otherwise}
\end{cases}\]

\[
\det(C_{[n-1],S}) = \begin{cases} 
\pm 1 & S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases}
\]
**Kirchhoff’s Matrix-Tree Theorem:**

\[ \forall i, \quad t(G) = \det(L_{i,i}) \]

all \( n \)-vertex trees: spanning trees of \( K_n \)

\[
L_{i,i} = \begin{bmatrix}
n - 1 & -1 & \cdots & -1 \\
-1 & n - 1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & n - 1
\end{bmatrix}
\]

**Cayley formula:**

\[
T_n = t(K_n) = \det(L_{i,i}) = n^{n-2}
\]