DEFORMATIONS OF FORMAL $\pi$-DIVISIBLE $\mathcal{O}$-MODULES VIA $\mathcal{O}$-DISPLAYS

CHUANGXUN CHENG

Abstract. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer $\pi$ and $R$ be an $\mathcal{O}$-algebra with $\pi$ nilpotent in $R$. In this paper, we study deformations of $\mathcal{O}$-displays over $R$ by explicit computation. Since the category of nilpotent $\mathcal{O}$-displays over $R$ is equivalent to the category of formal $\pi$-divisible $\mathcal{O}$-modules over $R$, we obtain results on deformations of formal $\pi$-divisible $\mathcal{O}$-modules, which generalize the corresponding results on formal $p$-divisible groups.

1. Introduction

The theory of displays, which was developed by Zink and Lau in a series of papers ([13, 14, 8, 9, 10] etc.), is a powerful tool in the study of $p$-divisible groups. One of the main results of this theory is a classification result, which says that, for any ring $R$ with $p$ nilpotent in it, the category of formal $p$-divisible groups over $R$ and the category of nilpotent displays over $R$ are equivalent. Moreover, if $R$ is a Noetherian local ring with perfect residue field of characteristic $p$, the category of $p$-divisible groups over $R$ and the category of Dieudonné displays over $R$ are equivalent.

The above classification result was generalized in [1, 2]. In particular, we have the following result, which is the starting point of this paper. Let $p > 2$ be a prime. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer $\pi$. Let $R$ be an $\mathcal{O}$-algebra with $\pi$ nilpotent in it. Denote by $\text{ndisp}_\mathcal{O}/R$ the category of nilpotent $\mathcal{O}$-displays over $R$. From [2, Theorem 1.1], there exists a covariant functor $\text{BT}_\mathcal{O}$

$$\text{BT}_\mathcal{O} : \text{ndisp}_\mathcal{O}/R \to (\pi\text{-divisible formal } \mathcal{O}\text{-modules}/R),$$

which is an equivalence of categories.

The classification results in [13, 14, 8, 9, 10] have many applications in the study of $p$-divisible groups. In [2, 3], the authors generalized the classification results and obtained several applications in the study of $\pi$-divisible $\mathcal{O}$-modules. A simple idea is that, a $\pi$-divisible $\mathcal{O}$-module $X$ is a $p$-divisible group with a special $\mathcal{O}$-action and this special action includes extra information of the structure of $X$. Hence if we confine our study in the category of $\pi$-divisible $\mathcal{O}$-modules, we should obtain stronger results than those regarding general $p$-divisible groups.

1Keywords: $\pi$-divisible $\mathcal{O}$-module, deformation of $\pi$-divisible $\mathcal{O}$-module, (nilpotent) $\mathcal{O}$-display, Dieudonné $\mathcal{O}$-display, Lubin-Tate group
MSC2010: 14L05, 11S31
Department of Mathematics, Nanjing University, Nanjing 210093, China
Email: cxcheng@nju.edu.cn
In this paper, following the idea in \[13\] Sections 2.2, 2.5, we study deformations of \(\mathcal{O}\)-displays by explicit computation. Then by \[2\] Theorem 1.1, we translate the properties of \(\mathcal{O}\)-displays to properties of \(\pi\)-divisible \(\mathcal{O}\)-modules. To state the main results, we first fix some notation.

Let \(p > 2\) be a prime. Let \(\mathcal{O}\) be the ring of integers of a finite extension of \(\mathbb{Q}_p\) with uniformizer \(\pi\) and residue field \(\mathbb{F} = \mathbb{F}_q\). The category of \(\mathcal{O}\)-algebras is denoted by \(\text{Alg}_\mathcal{O}\). For \(A \in \text{Alg}_\mathcal{O}\), \(W_\mathcal{O}(A)\) is the ring of ramified Witt vectors. The Frobenius and Verschiebung morphisms on \(W_\mathcal{O}(A)\) are denoted by \(F\) and \(V\). The Teichmüller lift of \(a \in A\) is denoted by \([a]\) \(\in W_\mathcal{O}(A)\). Denote by \(I_\mathcal{O}(A)\) the image of the Verschiebung, i.e., \(I_\mathcal{O}(A) = VW_\mathcal{O}(A)\). See \[2\] Section 1.2.1 for more details.

For a \(\pi\)-divisible \(\mathcal{O}\)-module \(X\), \(X[\pi^n]\) denotes the \(\pi^n\)-torsion of \(X\). If \(X\) is of height \(h\) and dimension \(d\), we say that \(X\) is of type \((h,d)\).

For \(\mathcal{O}\)-displays and \(\mathcal{O}\)-windows, we will use without comment the notation of \[2\] \[3\]. For an \(\mathcal{O}\)-display \(\mathcal{P} = (P,Q,F,F_1)\) over \(R \in \text{Alg}_\mathcal{O}\), we say that \(\mathcal{P}\) is of type \((h,d)\) if \(P\) is free of rank \(h\) over \(W_\mathcal{O}(R)\) and \(P/Q\) is free of rank \(d\) over \(R\).

We prove the following results, which are well-known for \(p\)-divisible groups (cf. \[7\] \[4\]).

**Theorem 1.1.** Let \(R \in \text{Alg}_\mathcal{O}\) such that \(\pi\) is nilpotent in \(R\).

1. Let \(X\) be a formal \(\pi\)-divisible \(\mathcal{O}\)-module over \(R\) with type \((h,d)\). The deformation functor \(\mathbb{D}_X\) (cf. Section 3.1) is pro-representable by a formal \(\pi\)-divisible \(\mathcal{O}\)-module over \(R[[t_1, \ldots, t_{d(h-d)}]]\).

2. Let \(X\) and \(Y\) be two formal \(\pi\)-divisible \(\mathcal{O}\)-modules over \(R\) with \(X[\pi^n] = Y[\pi^n]\) for a positive integer \(n\). Let \(\tilde{X}\) be a deformation of \(X\) over \(S \in \text{Aug}_R\) (cf. Section 2.2). Then there exists a deformation \(\tilde{Y}\) of \(Y\) over \(S\) such that \(\tilde{Y}[\pi^n] \cong \tilde{X}[\pi^n]\).

**Remark 1.2.** If \(R = k \in \text{Alg}_\mathcal{O}\) is a perfect field of characteristic \(p\), then using \[2\] Theorem 1.5] and the theory of Dieudonné \(\mathcal{O}\)-displays, the same argument in this paper proves the following claims in equal-characteristic case.

1. Let \(X\) be a \(\pi\)-divisible \(\mathcal{O}\)-module over \(k\) with type \((h,d)\). The deformation functor \(\mathbb{D}_X\) is pro-representable by a \(\pi\)-divisible \(\mathcal{O}\)-module over \(k[[t_1, \ldots, t_{d(h-d)}]]\).

2. Let \(X\) and \(Y\) be two \(\pi\)-divisible \(\mathcal{O}\)-modules over \(k\) with \(X[\pi^n] = Y[\pi^n]\) for a positive integer \(n\). Let \(\tilde{X}\) be a deformation of \(X\) over \(S \in \text{Aug}_k\). Then there exists a deformation \(\tilde{Y}\) of \(Y\) over \(S\) such that \(\tilde{Y}[\pi^n] \cong \tilde{X}[\pi^n]\).

Let \((\mathcal{O}', \pi')\) be a totally ramified extension of \((\mathcal{O}, \pi)\) with degree \(e\). Let \(\tilde{X}\) over \(\mathcal{O}^u = W_{\mathcal{O}'}(\mathbb{F})\) be the \(\pi'\)-divisible Lubin-Tate group associated with the \(\mathcal{O}'\)-display
\[
(W_{\mathcal{O}'}(W_{\mathcal{O}'}(\mathbb{F})), I_{\mathcal{O}'}(W_{\mathcal{O}'}(\mathbb{F})), F, V^{-1}).
\]
Let \(X\) over \(\mathbb{F}\) be the \(\pi'\)-divisible Lubin-Tate group associated with the \(\mathcal{O}'\)-display
\[
(W_{\mathcal{O}'}(\mathbb{F}), I_{\mathcal{O}'}(\mathbb{F}), F, V^{-1}).
\]
Then \(X = \tilde{X} \otimes \mathbb{F}\) and is a formal \(\pi\)-divisible \(\mathcal{O}\)-module over \(\mathbb{F}\) with a special \(\mathcal{O}'\)-action (cf. \[2\] Section 1.2.3). As a formal \(\pi\)-divisible \(\mathcal{O}\)-module, the endomorphism ring \(\text{End}(X) = \mathcal{O}_D\), where \(D\) is the central simple \(\text{Frac}(\mathcal{O})\)-algebra with invariant \(1/e\) and \(\mathcal{O}_D\) is the maximal order of \(D\). Let \(X_m\) be the base change \(\tilde{X} \otimes_{\mathcal{O}^u} \mathcal{O}^u/(\pi')^{m+1}\). Then we have the following result, which may be considered as a relative version of a result of Gross (cf. \[6\] and \[13\] Proposition 79)).
**Theorem 1.3.** With the notation as above, we have 
\[ \text{End}(X_m) = \mathcal{O}' + (\pi')^m \mathcal{O}_D, \]
for all \( m \in \mathbb{Z}_{\geq 0} \).

### 2. Deformations of \( \mathcal{O} \)-displays

In this section, we study deformations of \( \mathcal{O} \)-displays and obstructions of lifting homomorphisms. In particular, we show that the deformation functor is pro-representable and describe the universal object explicitly. Since we are interested in nilpotent objects, the \( \mathcal{O} \)-displays in the rest of this paper are all assumed to be nilpotent without further comment.

#### 2.1. Liftings of an \( \mathcal{O} \)-display

Let \( R \) be an \( \mathcal{O} \)-algebra. Let \( \mathcal{P} \) be an \( \mathcal{O} \)-display over \( R \). Let \( S \to R \) be a surjection of \( \mathcal{O} \)-algebras. A lifting of \( \mathcal{P} \) to \( S \) is an \( \mathcal{O} \)-display \( \mathcal{P}' \) over \( S \) such that the base change of \( \mathcal{P}' \) with respect to \( S \to R \) is isomorphic to \( \mathcal{P} \). It is known that to lift \( \mathcal{P} \) to \( S \) is equivalent to lifting the Hodge filtration (cf. [13, Lemma 2.18])

\[ \text{Fil}^1_p(R) := Q/I_\mathcal{O}(R)P \subset \text{Fil}_p(R) := P/I_\mathcal{O}(R)P. \]

Note that this is denoted by \( \mathcal{D}_p(R) \subset \mathcal{D}_p(R) \) in [13].

Let us consider the special case, where \( S \to R \) is a surjection with kernel \( a \), such that \( a^2 = 0 \). Define an abelian group \( \mathcal{G} \) by

\[ \mathcal{G} := \text{Hom}(\text{Fil}^1_p(R), a \otimes_R (\text{Fil}_p(R)/\text{Fil}^1_p(R))). \]

We define an action of \( \mathcal{G} \) on the set of liftings of \( \mathcal{P} \) to \( S \) as follows. Two liftings of \( \mathcal{P} \) to \( S \) correspond to two liftings \( E_1 \) and \( E_2 \) of the Hodge filtration, i.e., \( E_1 \) and \( E_2 \) are both direct summand of \( \text{Fil}_p(S) \) that lifts \( \text{Fil}^1_p(R) \). Consider the natural homomorphism

\[ E_1 \subset \text{Fil}_p(S) \to \text{Fil}_p(S)/E_2. \]

Since \( E_1 \equiv E_2 \pmod{a} \), the homomorphism (2.2) factors as

\[ E_1 \to a(\text{Fil}_p(S)/E_2) \subset \text{Fil}_p(S)/E_2. \]

Moreover, since \( a^2 = 0 \), we have an isomorphism \( a(\text{Fil}_p(S)/E_2) \cong a \otimes_R (\text{Fil}_p(R)/\text{Fil}^1_p(R)). \)

Hence we obtain a homomorphism

\[ u : \text{Fil}^1_p(R) \to a \otimes_R (\text{Fil}_p(R)/\text{Fil}^1_p(R)). \]

Define \( E_1 - E_2 = u \). It is easy to check from the construction that

\[ E_2 = \{ e - \tilde{u}(e) \mid e \in E_1 \}, \]

where \( \tilde{u}(e) \in a \text{Fil}_p(S) \) denotes any lifting of \( u(e) \). We have the following result (cf. [13, Corollary 49]).

**Proposition 2.1.** Let \( \mathcal{P} \) be an \( \mathcal{O} \)-display over \( R \). Let \( S \to R \) be a surjection with kernel \( a \) such that \( a^2 = 0 \). The action of \( \mathcal{G} \) on the set of liftings of \( \mathcal{P} \) to \( S \) constructed as above is simply transitive. If \( \mathcal{P}_0 \) is a lifting of \( \mathcal{P} \) and \( u \in \mathcal{G} \), we denote the action by \( \mathcal{P}_0 + u \).

**Proof.** The transitivity follows from the construction. Moreover, if \( E_1 = E_2 \), then the object \( u \) constructed above is trivial. Hence the action is simple. The proposition follows. \( \square \)
Remark 2.2. The above action could be described more explicitly. Consider \( a \) as an ideal of \( W_\Omega(\mathfrak{a}) \) and we equip \( \mathfrak{a} \) with the trivial divided \( \Omega \)-pd-structure (cf. [3, Section 2.8]). Let \( \mathcal{P}_0 = (P_0, Q_0, F, F_1) \) be a lifting of \( \mathcal{P} \) to \( S \). Let \( \alpha : P_0 \to aP_0 \subset W_\Omega(\mathfrak{a})P_0 \) be a homomorphism. For the pair \((P_0, Q_0)\), we define a new \( \Omega \)-display structure by setting
\[
F_\alpha x = Fx - \alpha(Fx) \quad \text{for} \quad x \in P_0,
\]
\[
F_1\alpha y = F_1y - \alpha(F_1y) \quad \text{for} \quad y \in Q_0.
\]
By Proposition [2.1] there is an element \( u \in \mathcal{G} \) such that \( \mathcal{P}_\alpha = \mathcal{P}_0 + u \). This \( u \) could be described as follows. We have a natural isomorphism \( aP_0 \cong \mathfrak{a} \otimes_R P/I_\Omega(R)P \). Hence the homomorphism \( \alpha \) factors uniquely through a morphism
\[
\tilde{\alpha} : P/I_\Omega(R)P \to \mathfrak{a} \otimes_R P/I_\Omega(R)P.
\]
Conversely, any such \( R \)-module homomorphism \( \tilde{\alpha} \) determines a unique \( \alpha \). Let \( u \in \mathcal{G} \) be the composite of
\[
Q/I_\Omega(R)P \subset P/I_\Omega(R)P \xrightarrow{\tilde{\alpha}} \mathfrak{a} \otimes_R P/I_\Omega(R)P \to \mathfrak{a} \otimes_R P/Q.
\]
Then it is easy to check that \( \mathcal{P}_\alpha = \mathcal{P}_0 + u \).

2.2. Deformations of an \( \Omega \)-display. Let \( \Lambda \) be a topological \( \Omega \)-algebra of the following type. The topology on \( \Lambda \) is given by a filtration of \( \Omega \)-ideals
\[
\Lambda = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n \supset \cdots,
\]
such that \( \mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_{i+j} \). We assume that \( \pi \) is nilpotent in \( \Lambda/\mathfrak{a}_1 \) and hence in any quotient \( \Lambda/\mathfrak{a}_i \). Let \( R \in \text{Alg}_\Omega \) with the discrete topology. Suppose we are given a continuous surjective homomorphism \( \varphi : \Lambda \to R \).

Let \( \text{Aug}_{\Lambda \to R} \) be the category of morphisms of discrete \( \Lambda \)-algebras \( \psi_S : S \to R \), such that \( \psi_S \) is surjective and has a nilpotent kernel. If \( \Lambda = R \), we denote this category simply by \( \text{Aug}_R \).

Let \( \text{Nil}_R \) be the category of nilpotent \( R \)-algebras. Let \( \mathcal{N} \in \text{Nil}_R \). We associated with \( \mathcal{N} \) an augmented \( R \)-algebra \( R[\mathcal{N}] \) as follows. As an \( R \)-module, \( R[\mathcal{N}] = R \oplus \mathcal{N} \). The multiplication is given by
\[
(r_1 + n_1)(r_2 + n_2) = (r_1r_2) + (r_1n_2 + r_2n_1 + n_1n_2) \quad \text{for all} \quad r_1, r_2 \in R \quad \text{and} \quad n_1, n_2 \in \mathcal{N}.
\]
Let \( M \) be an \( R \)-module. We regard \( M \) as an object in \( \text{Nil}_R \) by setting \( M^2 = 0 \). Hence we obtain fully faithful functors \( \text{Mod}_R \subset \text{Nil}_R \subset \text{Aug}_{\Lambda \to R} \).

Definition 2.3. Let \( F \) be a set-valued functor on \( \text{Aug}_{\Lambda \to R} \). The restriction of this functor to the category of \( R \)-modules is denoted by \( t_F \) and is called the tangent functor of \( F \).

Definition 2.4. Let \( \mathcal{P} \) be an \( \Omega \)-display over \( R \). Let \( S \to R \) be a surjection of \( \Omega \)-algebras such that the kernel is nilpotent. A deformation of \( \mathcal{P} \) to \( S \) is an isomorphism class of pairs \( (\mathcal{P}', \iota) \), where \( \mathcal{P}' \) is an \( \Omega \)-display over \( S \) and \( \iota : \mathcal{P} \to \mathcal{P}'_R \) is an isomorphism. Here \( \mathcal{P}_R \) is the base change of \( \mathcal{P}' \) with respect to \( S \to R \) (cf. [2, Section 2.2]).

The deformation functor of \( \mathcal{P} \) is defined by
\[
\mathbb{D}_\mathcal{P} : \text{Aug}_{\Lambda \to R} \to \text{Sets}
\]
\[
S \mapsto \{\text{deformations of } \mathcal{P} \text{ to } S\}.
\]
We show that the functor $\mathbb{D}_P$ is pro-representable and construct the universal object. First we compute the tangent functor of $\mathbb{D}_P$. Let $M$ be an $R$-module. We study the liftings of $P$ to $R[M]$ with respect to the canonical map $R[M] \to R$. In this case, the kernel of $R[M] \to R$ is square-zero, we may apply Proposition 2.1 to this situation. In particular, we have an isomorphism:

$$\text{Hom}_R(Q/I_\mathcal{O}(R)P, M \otimes_R P/Q) \to \mathbb{D}_P(R[M]).$$

Note that in this case, we have a canonical choice for $P_0 = P_{R[M]}$ (cf. Remark 2.2). The tangent space of the functor $\mathbb{D}_P$ is isomorphic to the finitely generated projective $R$-module $\text{Hom}_R(Q/I_\mathcal{O}(R)P, P/Q)$. Define $\omega = \text{Hom}_R(P/Q, Q/I_\mathcal{O}(R)P)$. Then we have an isomorphism

$$\text{Hom}_R(\omega, M) \to \mathbb{D}_P(R[M]).$$

The identical endomorphism of $\omega$ defines a morphism of functors

$$(2.8) \quad \text{Spf } R|\omega| \to \mathbb{D}_P.$$

Let $\tilde{\omega}$ be a finitely generated projective $\Lambda$-module with $\tilde{\omega} \otimes_\Lambda R \cong \omega$. Let $S_\Lambda(\tilde{\omega})$ be the symmetric algebra. Let $A$ be the completion of the augmented algebra $S_\Lambda(\tilde{\omega})$ with respect to the augmentation ideal. The morphism $\omega$ may be lifted to a morphism

$$(2.9) \quad \text{Spf } A \to \mathbb{D}_P.$$

By our construction, the morphism (2.9) induces an isomorphism on the tangent spaces. Hence it is an isomorphism. Now we could describe the universal $\mathcal{O}$-display $\mathcal{P}_{\text{univ}}$ as follows. Let $u : Q/I_\mathcal{O}(R)P \to \omega \otimes_R P/Q$ be the map induced by the identical endomorphism of $\omega$. Let $\alpha : P \to \omega \otimes_R P/Q$ be any map that induces $u$ (cf. Remark 2.2). Then we obtain an $\mathcal{O}$-display $\mathcal{P}_\alpha$ over $R|\omega|$. Lifting $\mathcal{P}_\alpha$ to $A$, we obtain $\mathcal{P}_{\text{univ}}$.

**Remark 2.5.** We may write down the universal object explicitly in terms of structure equation as follows (cf. [12 Section (1.12)] and [13 Equation (87)]). Assume that $\mathcal{P} = (P, Q, F, F_1)$ and $P = L \oplus T$ is a normal decomposition of $\mathcal{P}$. Then $\mathcal{P}$ is determined by its structure equation

$$\Phi := F_1 \oplus F : L \oplus T \to P.$$

Here $F_1 \oplus F$ is an $F$-linear isomorphism. Assume further that $L$ and $T$ are finitely generated free $W_\mathcal{O}(R)$-modules, which is automatic if $W_\mathcal{O}(R)$ is local. Assume that the rank of $L$ is $c$ and the rank of $T$ is $d$. Fix a basis of $L$ and $T$, hence a basis of $P$, $F_1 \oplus F$ is given by a matrix $M_\mathcal{P} \in \text{GL}_h(W_\mathcal{O}(R)).$ Here $h = c + d$. We choose indeterminates $\{t_{ij} \mid 1 \leq i \leq c, 1 \leq j \leq d\}$ and set $A = \Lambda[[t_{ij}]]$. Define an invertible matrix in $\text{GL}_h(W_\mathcal{O}(A))$ by

$$\begin{pmatrix} \text{id}_c & [t_{ij}] \\ 0 & \text{id}_d \end{pmatrix} \tilde{M}_\mathcal{P}.$$

Here $\tilde{M}_\mathcal{P}$ is a lifting of $M_\mathcal{P}$ in $\text{GL}_h(W_\mathcal{O}(A))$ and $[t_{ij}]$ is the Teichmüller representative of $t_{ij}$. This matrix defines an $\mathcal{O}$-display $\mathcal{P}_{\text{univ}}$ over the topological ring $A$. Then the pair $(A, \mathcal{P}_{\text{univ}})$ pro-represents the functor $\mathbb{D}_\mathcal{P}$ on the category $\text{Aug}_{A \to R}$.

We could also see the meaning of $t_1, \ldots, t_{cd}$ in Remark 2.5 explicitly when we consider the infinitesimal deformations, i.e., deformations over the dual numbers $R[\epsilon] = R[x]/(x^2)$. 

Lemma 2.6. Let $\mathcal{P} = (P, Q, F, F_1)$ and $\mathcal{P}' = (P', Q', F, F_1)$ be two $\mathcal{O}$-displays over $R$. Then we have an exact sequence

$$0 \to \text{Hom}_{\mathcal{F}; \mathcal{F}1}(P, P') \to \text{Hom}_{\mathcal{F}}(P, P') \to \text{Ext}^1(\mathcal{P}, \mathcal{P}') \to 0. \quad (2.10)$$

Here $\text{Hom}_{\mathcal{F}}(P, P')$ means $F$-linear maps $P \to P'$, $\text{Hom}_{\mathcal{F}; \mathcal{F}1}(P, P')$ means $F$-linear maps $P \to P'$ that send $Q$ to $Q'$, and the second arrow is given by $\beta \mapsto (\beta \Phi^P - \Phi^{P'} \beta)$.

Proof. The proof is standard. Assume that we have a short exact sequence of $\mathcal{O}$-displays

$$0 \to \mathcal{P}' \to \mathcal{P}'' = (P'', Q'', F, F_1) \to \mathcal{P} \to 0.$$

We may write $P'' = P \oplus P'$ and $Q'' = Q \oplus Q'$. Choose normal decompositions of $\mathcal{P}$ and $\mathcal{P}'$, say $P = L \oplus T$ and $P' = L' \oplus T'$. Then $\mathcal{P}''$ is determined by the structure equation $F_1 \oplus F : (L \oplus L') \oplus (T \oplus T') \to (P \oplus P')$, which may be written as

$$F_1 \oplus F = \begin{pmatrix} F_1 \oplus F & \alpha \\ 0 & F_1 \oplus F \end{pmatrix},$$

where $\alpha \in \text{Hom}_F(P, P')$. Conversely, any element $\alpha \in \text{Hom}_F(P, P')$ gives rise to an extension of $\mathcal{O}$-displays. Moreover, two elements $\alpha$ and $\alpha'$ give rise to isomorphic extensions if there exists an element $\beta \in \text{Hom}_{F; \mathcal{F}1}(P, P')$ such that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \oplus F & \alpha \\ 0 & F_1 \oplus F \end{pmatrix} = \begin{pmatrix} F_1 \oplus F & \alpha' \\ 0 & F_1 \oplus F \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Hence the lemma follows. \qed

Corollary 2.7. Let $\mathcal{P}$ be an $\mathcal{O}$-display over $R$ of type $(h, d)$. Then

$$\text{Rank}_{\text{Alg}_R} \text{Ext}^1(\mathcal{P}, \mathcal{P}) = \text{Rank}_{\text{Alg}_R} \mathcal{D}_{\mathcal{P}}(R[\epsilon]) = d(h - d).$$

2.3. Lifting homomorphisms: part one. Let $\mathcal{P} = (P, Q, F, F_1)$ and $\mathcal{P}' = (P', Q', F, F_1)$ be two $\mathcal{O}$-displays over $R$. Let $S \to R$ be a surjection with nilpotent kernel $\eta$. Let $\mathcal{P} = (P, Q, F, F_1)$ be a lifting of $\mathcal{P}$ to $S$. Assume that there exists a homomorphism of $\mathcal{O}$-displays

$$\bar{f} : (\bar{P}, \bar{Q}, F, F_1) \to (P', Q', F, F_1).$$

Then we have the following result.

Proposition 2.8. With the notation as above. There exists a lifting $\mathcal{P}' = (P', Q', F, F_1)$ of $\mathcal{P}'$ to $S$ and a homomorphism

$$f : (P, Q, F, F_1) \to (P', Q', F, F_1),$$

such that $f$ lifts $\bar{f}$.

Proof. Since a homomorphism $\alpha : X \to Y$ could be encoded by the automorphism $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ on $X \oplus Y$, to prove the proposition, we may assume that $\bar{f}$ is an automorphism. Moreover, every nilpotent $\mathcal{N} \in \text{Alg}_R$ admits a filtration

$$\mathcal{N} = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_m \supset \mathcal{N}_{m+1} = 0,$$
such that $N_i^2 \subset N_{i+1} \ (0 \leq i \leq m)$. Hence we may assume that $a^2 = 0$. Therefore, the proposition follows from the following lemma.

**Lemma 2.9.** Let $\mathcal{P} = (P, Q, F, F_1)$ be a lifting of $\bar{\mathcal{P}} = (\bar{P}, \bar{Q}, F, F_1)$ from $R$ to $S = R[N]$ with $N^2 = 0$. Let $\bar{f}$ be an automorphism of $\bar{\mathcal{P}}$. Then there exists another lifting $\mathcal{P}' = (P', Q', F', F'_1)$ of $\bar{\mathcal{P}}$ to $S$ and an isomorphism

$$f : (P, Q, F, F_1) \to (P', Q', F, F'_1),$$

such that $f$ lifts $\bar{f}$.

**Proof.** Assume that $\mathcal{P}$ is of type $(h, d)$. We fix a normal decomposition $\bar{\mathcal{P}} = \bar{L} \oplus \bar{T}$ of $\bar{\mathcal{P}}$ and a basis for both $\bar{L}$ and $\bar{T}$. The structure of $\mathcal{P}$ is determined by a matrix $\Phi \in \text{GL}_h(W_\mathcal{O}(R))$, which corresponds to the $F$-linear isomorphism $F_1 \oplus F : \bar{L} \oplus \bar{T} \to \bar{P}$. The automorphism $\bar{f}$ corresponds to a matrix $X \in \text{GL}_h(W_\mathcal{O}(R))$, such that $X$ sends $\bar{L} \oplus I_\mathcal{O}(R)\bar{T}$ into $L \oplus I_\mathcal{O}(R)\bar{T}$. The structure of $\mathcal{P}$ corresponds to a matrix $\Phi + \Phi' \in \text{GL}_h(W_\mathcal{O}(S))$. Here we consider $\Phi$ as a matrix in $\text{GL}_h(W_\mathcal{O}(S))$ via the natural embedding $W_\mathcal{O}(R) \hookrightarrow W_\mathcal{O}(S)$, $\Phi_\mathcal{N}$ is a matrix in $M_h(W_\mathcal{O}(\mathcal{N}))$.

Finding the pair $(\mathcal{P}', f)$ is equivalent to finding matrices $\Phi_\mathcal{N}' \in M_h(W_\mathcal{O}(\mathcal{N}))$ and $X_\mathcal{N} \in M_h(W_\mathcal{O}(\mathcal{N}))$ with the property

$$\Phi + \Phi_\mathcal{N}'(X + X_\mathcal{N}) = (X + X_\mathcal{N})(\Phi + \Phi_\mathcal{N}),$$

because then we may take $\mathcal{P}'$ to be the $\mathcal{O}$-display with structure equation given by $\Phi + \Phi_\mathcal{N}'$, $f$ to be the homomorphism given by $X + X_\mathcal{N}$.

Note that $\Phi X = X\Phi$ since $X$ induces a homomorphism of $\mathcal{O}$-displays. Define

$$\Phi_\mathcal{N}' = \Phi X_\mathcal{N}\Phi^{-1}X^{-1},$$

$$X_\mathcal{N} = -X\Phi_\mathcal{N}\Phi^{-1} = -\Phi^{-1}\Phi_\mathcal{N}'X.$$

Since $N^2 = 0$, we have $\Phi_\mathcal{N}'X_\mathcal{N} = X_\mathcal{N}\Phi_\mathcal{N}' = 0$. It is easy to check that

$$\Phi X_\mathcal{N} - X_\mathcal{N}\Phi = -\Phi_\mathcal{N}'X + X\Phi_\mathcal{N},$$

The pair $(\Phi_\mathcal{N}', X_\mathcal{N})$ defined by equation $2.12$ satisfies equation $2.11$. The lemma follows.

By the same discussion as above, we have the following result.

**Proposition 2.10.** Let $\bar{\mathcal{P}} = (\bar{P}, \bar{Q}, F, F_1)$ and $\bar{\mathcal{P}}' = (\bar{P}', \bar{Q}', F, F_1)$ be two $\mathcal{O}$-displays over $R$. Let $S \to R$ be a surjection with nilpotent kernel. Let $\mathcal{P} = (P, Q, F, F_1)$ be a lifting of $\bar{\mathcal{P}}$ to $S$. Assume that there exists a homomorphism between quadruples

$$\bar{f} : (\bar{P}/\pi^n, \bar{Q}/\pi^n, F, F_1) \to (\bar{P}'/\pi^n, \bar{Q}'/\pi^n, F, F_1)$$

for some $n \in \mathbb{Z}_{\geq 0}$. Then there exists a lifting $\mathcal{P}' = (P', Q', F, F_1)$ of $\bar{\mathcal{P}}'$ to $S$ and a homomorphism

$$f : (P/\pi^n, Q/\pi^n, F, F_1) \to (P'/\pi^n, Q'/\pi^n, F, F_1),$$

such that $f$ lifts $\bar{f}$. 
2.4. Lifting homomorphisms: part two. In Section 2.3, we saw that liftings of a homomorphism \( f : \mathcal{P}_1 \to \mathcal{P}_2 \) always exist if we are allowed to change the liftings of the \( \mathcal{O} \)-displays. The situation changes completely if we fix the liftings of the \( \mathcal{O} \)-displays, as we shall see in this section.

Let \( S \to R \) be an \( \mathcal{O} \)-pd-thickening with kernel \( a \). Assume that \( \pi \) is nilpotent in \( S \). Let \( \mathcal{P}_i = (P_i, Q_i, F, F_i) \) \((i = 1, 2)\) be two \( \mathcal{O} \)-displays over \( S \). Denote by \( \mathcal{P}_i = (P_i, Q_i, F, F_i) \) the base change of \( \mathcal{P}_1 \) to \( R \). Let \( \bar{\varphi} : \mathcal{P}_1 \to \mathcal{P}_2 \) be a morphism of \( \mathcal{O} \)-displays. It lifts to a morphism of \( \mathcal{O} \)-windows over \( W_{S/R} \) (cf. [3, Section 2.8])

\[
\varphi : (P_1, \hat{Q}_1, F, F_1) \to (P_2, \hat{Q}_2, F, F_1).
\]

Note that in [13, Section 2.5], Zink used \( \mathcal{P} \)-triples, which are the same as \( \mathcal{O} \)-windows over \( W_{S/R} \). The morphism \( \varphi \) does not induce a morphism from \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \) in general. We may describe the obstruction as follows. Consider the composition

\[
\text{Obst} \bar{\varphi} : Q_1/I_\mathcal{O}(S)P_1 \to P_1/I_\mathcal{O}(S)P_1 \xrightarrow{\bar{\varphi}} P_2/I_\mathcal{O}(S)P_2 \to P_2/Q_2.
\]

Since \( \bar{\varphi}(Q_1) \subset \hat{Q}_2 \), \( \text{Obst} \bar{\varphi} \) is trivial modulo \( a \). Hence we obtain a map

\[
\text{Obst} \bar{\varphi} : Q_1/I_\mathcal{O}(S)P_1 \to a \otimes_S P_2/Q_2,
\]

which is zero if and only if \( \bar{\varphi} \) lifts to a morphism of \( \mathcal{O} \)-displays \( \mathcal{P}_1 \to \mathcal{P}_2 \) (i.e., \( \varphi \) sends \( Q_1 \) into \( Q_2 \)). We call it the obstruction to lift \( \bar{\varphi} \) to \( S \).

Remark 2.11. The obstruction has functorial property. Assume that we have a morphism \( \alpha : \mathcal{P}_2 \to \mathcal{P}_3 \) of \( \mathcal{O} \)-displays over \( S \). Let \( \bar{\alpha} : \mathcal{P}_2 \to \mathcal{P}_3 \) be its reduction over \( R \). Then \( \text{Obst} \bar{\alpha} \circ \bar{\varphi} \) is the composite of the following maps

\[
Q_1/I_\mathcal{O}(S)P_1 \xrightarrow{\text{Obst} \bar{\varphi}} a \otimes_S P_2/Q_2 \xrightarrow{1 \otimes \alpha} a \otimes_S P_3/Q_3.
\]

We denote this fact by

\[
\text{Obst} \bar{\alpha} \bar{\varphi} = \alpha \text{Obst} \bar{\varphi}.
\]

Remark 2.12. In the case \( a^2 = 0 \), we have \( a \otimes_S P_2/Q_2 \cong a \otimes_R \hat{P}_2/\hat{Q}_2 \). In this case, the obstruction may be considered as a map

\[
\text{Obst} \bar{\varphi} : \hat{Q}_1/I_\mathcal{O}(R)\hat{P}_1 \to a \otimes_R \hat{P}_2/\hat{Q}_2.
\]

This is compatible with Proposition 2.1. Equation (2.16) may be written as

\[
\text{Obst} \bar{\alpha} \bar{\varphi} = \bar{\alpha} \text{Obst} \bar{\varphi}.
\]

Let \( S \) and \( \hat{S} \) be \( \mathcal{O} \)-algebras such that \( \pi S = \pi \hat{S} = 0 \). Let \( S \to R \) be a surjection with kernel \( a \) such that \( a^2 = 0 \). Let \( \hat{S} \to S \) be a surjection with kernel \( b \) such that \( b^2 = 0 \). We equip \( a \) and \( b \) with the trivial \( \mathcal{O} \)-structure, hence \( S \to R \) and \( \hat{S} \to S \) are both \( \mathcal{O} \)-pd-thickenings.

Assume that \( \mathcal{P}_i \) is the base change of an \( \mathcal{O} \)-display \( \mathcal{P}_i \) over \( \hat{S} \) with respect to \( \hat{S} \to S \) \((i = 1, 2)\). Consider \( \pi \bar{\varphi} : \mathcal{P}_1 \to \mathcal{P}_2 \), a morphism of \( \mathcal{O} \)-displays over \( R \). It lifts to a morphism

\[
\pi \varphi : (P_1, \hat{Q}_1, F, F_1) \to (P_2, \hat{Q}_2, F, F_1).
\]

This morphism induces a morphism \( \pi \varphi : \mathcal{P}_1 \to \mathcal{P}_2 \), as \( \text{Obst} \pi \bar{\varphi} \) is trivial.
Remark 2.13. The morphism $\bar{\varphi} : \bar{P}_1 \to \bar{P}_2$ also lifts to a morphism

$$\varphi : (P_1, \bar{Q}_1, F, F_1) \to (P_2, \bar{Q}_2, F, F_1).$$

But $\varphi$ does not induce a morphism from $P_1$ to $P_2$ in general. On the other hand, $\pi \cdot \varphi$ does as $\pi a = 0$ and the obstruction vanishes.

In the following, we study the obstruction to lift $\pi \varphi$ to a homomorphism of $O$-displays $\tilde{P}_1 \to \tilde{P}_2$, i.e., the map

$$\text{Obst} \pi \varphi : \tilde{Q}_1/I_O(\tilde{S})\tilde{P}_1 \to b \otimes \tilde{P}_2/\tilde{Q}_2.$$  

The obstruction $\text{Obst} \pi \varphi$ may be computed in terms of $\text{Obst} \bar{\varphi}$. In order to do so, we need to define two other maps.

**The map $V^\sharp$:** The image of $F_1 : \tilde{Q}_1 \to \tilde{P}_1$ generates $\tilde{P}_1$, hence it induces a surjection

$$F_1^\sharp : \tilde{S} \otimes_{S, \text{Frob}} \tilde{Q}_1/I_O(\tilde{S})\tilde{P}_1 \to \tilde{P}_1/(I_O(\tilde{S})\tilde{P}_1 + W_O(\tilde{S})\tilde{P}_1).$$

Using the normal decomposition of $\tilde{P}_1$, one sees that the left hand side and the right hand side are projective $\tilde{S}$-modules of the same rank. Hence $F_1^\sharp$ is an isomorphism. Let $V^\sharp$ be the inverse of $F_1^\sharp$. Note that $b$ is in the kernel of the Frobenius morphism, we have an isomorphism

$$\tilde{S} \otimes_{S, \text{Frob}} \tilde{Q}_1/I_O(\tilde{S})\tilde{P}_1 \cong \tilde{S} \otimes_{S, \text{Frob}} Q_1/I_O(S)P_1.$$

It induces the following map, which we still denote by $V^\sharp$

$$V^\sharp : \tilde{P}_1 \to \tilde{S} \otimes_{S, \text{Frob}} Q_1/I_O(S)P_1.$$

**The map $F^\sharp$:** We have assumed that $b^a = 0$, so the operator $F$ on $\tilde{P}_2/I_O(\tilde{S})\tilde{P}_2$ factors as

$$\begin{array}{ccc}
\tilde{P}_2/I_O(\tilde{S})\tilde{P}_2 & \xrightarrow{F} & \tilde{P}_2/I_O(\tilde{S})\tilde{P}_2 \\
\downarrow \quad & & \quad \downarrow F^b \\
P_2/I_O(S)P_2 & \quad & \tilde{P}_2/I_O(\tilde{S})\tilde{P}_2
\end{array}$$

Moreover, from the definition of $O$-displays, $F(x) = \pi F_1(x)$ if $x \in \tilde{Q}_2$. Hence $\tilde{Q}_2/I_O(\tilde{S})\tilde{P}_2 \in \text{Ker}(F)$ and we obtain a Frobenius-linear map

$$F^b : P_2/Q_2 \to \tilde{P}_2/I_O(\tilde{S})\tilde{P}_2.$$ 

Restricting $F^b$ to $a(P_2/Q_2)$, we obtain

$$F^b : a(P_2/Q_2) \to b(\tilde{P}_2/I_O(\tilde{S})\tilde{P}_2).$$

Note that we may view $b$ as an ideal of $W_O(b)$ (cf. [3, Section 2.8]). Hence we may and do identify $b(\tilde{P}_2/I_O(\tilde{S})\tilde{P}_2)$ with $b\tilde{P}_2$. Denote by $F^\sharp$ the linearization of $F^b$

$$F^\sharp : \tilde{S} \otimes_{S, \text{Frob}} a(P_2/Q_2) \to b\tilde{P}_2.$$
Proposition 2.14. The following diagram is commutative

\[
\begin{array}{c}
\hat{Q}_1/I_O(\hat{S})\hat{P}_1 \xrightarrow{\psi} \hat{S} \otimes_{S,\text{Frob}} Q_1/I_O(S)P_1 \xrightarrow{\hat{S} \otimes \text{Obst}\varphi} \hat{S} \otimes_{S,\text{Frob}} a(P_2/Q_2) \\
\text{Obst}(\pi\varphi) \downarrow \downarrow \downarrow F^2 \\
\hat{b}(\hat{P}_2/\hat{Q}_2)
\end{array}
\]

Sketch of the proof. The morphism of \(O\)-displays \(\pi\varphi : P_1 \to P_2\) lifts to a uniquely determined morphism of \(O\)-windows over \(W_{\hat{S}/S}\)

\[
\hat{\psi} : (\hat{P}_1, \hat{Q}_1, F, F_1) \to (\hat{P}_2, \hat{Q}_2, F, F_1).
\]

Let \(\hat{\varphi} : \hat{P}_1 \to \hat{P}_2\) be any \(W_O(\hat{S})\)-linear map that lifts \(\varphi : P_1 \to P_2\) (cf. Remark 2.13). It does not induce a morphism \(\hat{P}_1 \to \hat{P}_2\) of \(O\)-windows over \(W_{\hat{S}}\) since it does not commute with \(F_1\) in general. On the other hand, we have

\[
\hat{\psi} = \pi\hat{\varphi} + \omega,
\]

where \(\omega : \hat{P}_1 \to \hat{b}\hat{P}_2 \subset W_O(b)\hat{P}_2\) is the composite of the following maps

\[
\hat{P}_1 \xrightarrow{\psi} \hat{S} \otimes_{S,\text{Frob}} Q_1/I_O(S)P_1 \xrightarrow{\hat{S} \otimes \text{Obst}\varphi} \hat{S} \otimes_{S,\text{Frob}} a(P_2/Q_2) \xrightarrow{F^2} \hat{b}\hat{P}_2.
\]

Equation (2.22) could be proven by the same argument of [13, Corollary 74], which is closely related to [13, Theorem 44] and [3, Theorem 2.12]. Then the proposition follows easily. \(\square\)

3. DEFORMATIONS OF FORMAL \(\pi\)-DIVISIBLE \(O\)-MODULES

In this section, we translate the results in Section 2 via [2, Theorem 1.1]. In particular, we obtain Theorem 1.1.

3.1. The universal deformation. Let \(R \in \text{Alg}_O\) with \(\pi\) nilpotent in it. Let \(X\) be a formal \(\pi\)-divisible \(O\)-module over \(R\). Let \(S \to R\) be a surjection with nilpotent kernel. A deformation of \(X\) to \(S\) is an isomorphism class of pairs \((X', \iota)\), where \(X'\) is a formal \(\pi\)-divisible \(O\)-module over \(S\) and \(\iota : X' \times_S R \cong X\) is an isomorphism of formal \(\pi\)-divisible \(O\)-modules. The deformation functor of \(X\) is defined by

\[
\mathbb{D}_X : \text{Aug}_{\Lambda} \to \text{Sets} \to \{\text{deformations of } X \text{ to } S\}.
\]

(3.1)

\[
\mathbb{D}_X(S) = \text{Hom}(\Lambda[[t_1, \cdots, t_{dc}]], S)
\]

and every deformation of \(X\) over \(S\) is a base change induced by a morphism in the left hand side of equation (3.2). Here \(c = h - d\) and \(X\) is of type \((h, d)\).
3.2. On the truncations. Let $R \in \text{Alg}_O$ with $\pi$ nilpotent in it. Let $X_1$ and $X_2$ be formal $\pi$-divisible $O$-modules over $R$.

**Theorem 3.2.** If $X_1[\pi^n] \cong X_2[\pi^n]$, then for any deformation $\tilde{X}_1$ of $X_1$ over $S$, there exists a deformation $\tilde{X}_2$ of $X_2$ over $S$, such that $\tilde{X}_1[\pi^n] \cong \tilde{X}_2[\pi^n]$.

**Proof.** Let $BT_{O,n}$ be the category of special truncated formal $\pi$-divisible $O$-modules with level $n$. (Here *special* means that the truncated $O$-modules are kernels of isogenies of formal $\pi$-divisible $O$-modules.) Then $BT_{O,n}$ is a smooth Artin algebraic stack with affine diagonal. The truncation morphism $BT_{O,n+1} \to BT_{O,n}$ is smooth and surjective by the same argument of [11, Proposition 3.15]. (See also [2, Lemma 4.4].) The theorem then follows from Proposition 2.8. 

**Remark 3.3.** This result was also indicated in [5, Section 8].

**Remark 3.4.** Let $X$ be a formal $\pi$-divisible $O$-module over $R$. Let $\mathcal{P} = (P,Q,F,F_1)$ be the corresponding $O$-display. Then by [2, Theorem 2.12], $X$ is determined by the following exact sequence

$$0 \to \hat{Q}_N \xrightarrow{\text{id} - F_1} \hat{P}_N \to X(\mathcal{N}) \to 0.$$ 

By Snake Lemma, $X[\pi^n]$ lies in the exact sequence

$$X[\pi^n](\mathcal{N}) \to \hat{Q}_N[\pi^n] \xrightarrow{\text{id} - F_1} \hat{P}_N[\pi^n].$$

If the first arrow is an injection, then $X[\pi^n]$ is determined by the quadruple $(P/\pi^n, Q/\pi^n, F, F_1)$ and the theorem follows from Proposition 2.10. In general, the first arrow has non-trivial kernel and we need to adapt to stacks $BT_{O,n}$ to prove our claim.

For formal $p$-divisible groups, Theorem 3.2 follows from [7, Théorème 4.4], which is proved by a different method.

3.3. A result of Keating. Let $k \in \text{Alg}_O$ be an algebraically closed field of characteristic $p$. Let $X_0$ be a $\pi$-divisible $O$-module of height 2 and dimension 1. Then $\text{End}(X_0)$ is the ring of integers in a quaternion algebra $D$ with center $\text{Frac}(O)$. Let $O_D = \text{End}(X_0)$. Let $\alpha \mapsto \alpha^*$ be the main involution on $O_D$. Fix $\alpha \in O_D$ such that $\alpha \not\in O$ and set $\iota = \text{ord}_O(\alpha - \alpha^*)$. Define $c(\alpha) \in \mathbb{N}$ by

$$c(\alpha) = \begin{cases} q^{\iota/2} + 2 \sum_{j=1}^{\lfloor \iota/2 \rfloor} q^{\iota/2 - j} & \text{if } 2 \mid \iota, \\ 2 \sum_{j=0}^{\lfloor \iota/2 \rfloor - 1} q^{\iota/2 - j} & \text{if } 2 \nmid \iota. \end{cases}$$

Let $X$ over $k[[t]]$ be the universal deformation of $X_0$ in equal characteristic.

**Theorem 3.5.** With the notation as above, $\alpha$ lifts to an endomorphism of $X \otimes_{k[[t]]} k[[t]]/t^{c(\alpha)}$ but does not lift to an endomorphism of $X \otimes_{k[[t]]} k[[t]]/t^{c(\alpha)+1}$.

**Proof.** If we translate the above statement on $\pi$-divisible $O$-modules to a statement on $O$-displays and use Proposition 2.14 the proof then goes entirely similar as the proof of [13, Proposition 75].

4. On Lubin-Tate groups

In this section, we study Lubin-Tate groups and prove Theorem 1.3. The main idea is to use the relation between $O'$-displays and $(O,O')$-displays, which is an essential ingredient in the proof of [2, Theorem 1.1].
4.1. The general set-up. Let $A$ be an $O$-algebra and $S$ be an $A$-algebra. An $(O, A)$-display over $S$ is a pair $(P, \iota)$, where $P$ is an $O$-display over $S$ and $\iota : A \to \text{End}(P)$ is a ring homomorphism, such that the action of $A$ on $P/Q$ induced from $\iota$ coincides with action from the structure morphism $A \to S$.

Let $a \in A$ be a fixed element. Set $R = S/a$ and $R_i = S/a^{i+1}$. Then we have a sequence of surjections
\[ S \to \cdots \to R_i \to R_{i-1} \to \cdots \to R_0 = R. \]

Let $\tilde{P}_1$ and $\tilde{P}_2$ be $O$-displays over $S$. By base change, we have $O$-displays $P^{(i)}_1$ and $P^{(i)}_2$ over $R_i$ for each $i \in \mathbb{Z}_{\geq 0}$. Set $P_1 = P^{(0)}_1$ and $P_2 = P^{(0)}_2$. Let $\varphi : P_1 \to P_2$ be a morphism of $O$-displays over $R$. Assume that $\varphi$ lifts to a morphism $\varphi^{(i-1)} : P^{(i-1)}_1 \to P^{(i-1)}_2$. To lift $\varphi^{(i-1)}$ to a morphism $P^{(i)}_1 \to P^{(i)}_2$ gives us the following obstruction morphism
\[ \text{Obst}(\varphi^{(i-1)} : Q_1^{(i)}/I_O(R_i)P^{(i)}_1 \to (a^i)/(a^{i+1}) \otimes_{R_i} P^{(i)}_2/Q^{(i)}_2, \]
which factors through (cf. Remark 2.12)
\[ \text{Obst}_i \varphi : Q_1/I_O(R)P_1 \to (a^i)/(a^{i+1}) \otimes_R P_2/Q_2. \]

Moreover, the obstruction to lift $\iota(a)\varphi^{(i-1)}$ to a morphism $P^{(i+1)}_1 \to P^{(i+1)}_2$ is given by
\[ \text{Obst}(\iota(a)\varphi^{(i-1)} : Q_1^{(i+1)}/I_O(R_{i+1})P^{(i+1)}_1 \to (a^{i+1})/(a^{i+2}) \otimes_{R_{i+1}} P^{(i+1)}_2/Q^{(i+1)}_2, \]
which factors through
\[ \text{Obst}_{i+1}(\iota(a)\varphi) : Q_1/I_O(R)P_1 \to (a^i)/(a^{i+2}) \otimes_R P_2/Q_2. \]

Since $\iota(a)$ acts on $P^{(i+1)}/Q^{(i+1)}$ by multiplication by $a$, we have the following commutative diagram
\[
\begin{array}{ccc}
Q_1/I_O(R)P_1 & \xrightarrow{\text{Obst}_i(\varphi)} & (a^i)/(a^{i+1}) \otimes_R P_2/Q_2 \\
\downarrow{\text{Obst}_{i+1}(\iota(a)\varphi)} & & \downarrow{a \otimes \text{id}} \\
(a^i)/(a^{i+2}) \otimes_R P_2/Q_2 \\
\end{array}
\]

Therefore, we have the following result.

Lemma 4.1. Let $\tilde{P}_1$ and $\tilde{P}_2$ be $O$-displays over $S$. By base change, we have $O$-displays $P^{(i)}_1$ and $P^{(i)}_2$ over $R_i$ for each $i \in \mathbb{Z}_{\geq 0}$. Set $P_1 = P^{(0)}_1$ and $P_2 = P^{(0)}_2$. Let $\varphi : P_1 \to P_2$ be a morphism of $O$-displays over $R$. Assume that $\varphi$ lifts to a morphism $\varphi^{(i-1)} : P^{(i-1)}_1 \to P^{(i-1)}_2$. Then $\iota(a)\varphi$ lifts to a morphism $\varphi^{(i)} : P^{(i)}_1 \to P^{(i)}_2$.

4.2. The Lubin-Tate $O$-display. Let $E'$ be a totally ramified extension of $E = \text{Frac}(O)$ with degree $e \geq 2$. Let $O'$ be the ring of integers of $E'$ and $\pi'$ be a uniformizer of $O'$. In the following, we study a particular $(O, O')$-display. Let $S$ be the $O'$-algebra $O' \otimes_O W(O)(\overline{F})$. Denote by $a$ the image of $\pi'$ in $S$.

Let $P = O' \otimes_O W(O)(S)$. It is a free $W(O)(S)$-module with basis $\{ (\pi')^i \otimes 1 \mid 0 \leq i \leq e-1 \}$. Hence it is a free $W(O)(S)$-module with basis
\[ 1 \otimes 1, \ (\pi')^i \otimes 1 - 1 \otimes [a^i] \text{ for } 1 \leq i \leq e-1. \]
Define

\[ T = \mathcal{W}_\mathcal{O}(S)(1 \otimes 1), \quad L = \mathcal{W}_\mathcal{O}(S)(\langle \pi^i \rangle \otimes 1 - 1 \otimes \langle a^i \rangle | 1 \leq i \leq e - 1) \].

Then \( P = L \oplus T \). Define \( Q = L \oplus \mathcal{I}_\mathcal{O}(S)T \). We define an \( \mathcal{O} \)-display structure on the pair \( (P, Q) \) by writing down the structure equation explicitly. More precisely (cf. [13, Pages 24-25]),

\[
\begin{align*}
F_1((\pi^i) \otimes 1 - 1 \otimes \langle a^i \rangle) &= \frac{\pi^i \otimes 1 - 1 \otimes \langle a^i \rangle}{\pi^i \otimes 1 - 1 \otimes \langle a^i \rangle} = \sum_{0 \leq k \leq \frac{i}{i-1}} (\pi^k \otimes \langle a^q \rangle), \\
F(1 \otimes 1) &= \frac{\tau - 1(\pi^e \otimes 1 - 1 \otimes \langle a^q \rangle)}{\pi^e \otimes 1 - 1 \otimes \langle a^q \rangle}.
\end{align*}
\]

Here \( \tau = \pi^{-1}(\pi^e \otimes 1 - 1 \otimes \langle a^q \rangle) \). It is a unit in \( \mathcal{O}' \otimes \mathcal{O} \mathcal{W}_\mathcal{O}(S) \) by [2, Lemma 2.24].

Let \( \hat{\mathcal{P}} = (\hat{\mathcal{P}}, Q, F, F_1) \) be the \( \mathcal{O} \)-display over \( S \) defined as above. Let \( \mathcal{P} = (P, Q, F, F_1) \) be the \( \mathcal{O} \)-display over \( R = S/aS = \mathbb{F} \) defined by base change. Then \( Q = \pi'P \) and

\[ F_1((\pi^i) = (\pi^e)^{i-1} \quad \text{for} \quad i \geq 1, \]

where \( \pi' = \pi' \otimes 1 \in \mathcal{O}' \otimes \mathcal{O} \mathcal{W}_\mathcal{O}(\mathbb{F}) \).

Let \( \varphi : \mathcal{P} \to \mathcal{P} \) be an endomorphism of \( \mathcal{P} \). The obstruction to lift \( \varphi \) to \( R_1 = S/a^2S \) is

\[ \text{Obst}_1(\varphi) : Q/I_\mathcal{O}(R)P \to (a)/(a^2) \otimes_R P/Q. \]

The endomorphism \( \varphi \) induces an endomorphism on \( P/Q \cong \mathbb{F} \), which is the multiplication of some element in \( \mathbb{F} \). Denote this element by \( \text{Lie} \varphi \). Let \( \sigma \) be the Frobenius endomorphism of \( \mathbb{F} \) given by \( x \mapsto x^q \).

**Lemma 4.2.** With the notation as above, we have the following commutative diagram

\[
\begin{array}{ccc}
Q/I_\mathcal{O}(R)P &=& Q/P \to P/P' = P/Q \sigma^{-1}(\text{Lie} \varphi - \text{Lie} \varphi) \\
\text{Obst}_1(\varphi) \downarrow &=& \downarrow a \\
&=& (a)/(a^2) \otimes_R P/Q.
\end{array}
\]

**Proof.** For simplicity, if \( x \in \mathcal{O}' \), we still denote by \( x \) for the image \( x \otimes 1 \in \mathcal{O}' \otimes \mathcal{O} \mathcal{W}_\mathcal{O}(\mathbb{F}) = P \). Write

\[
\varphi(1) = \xi_0 + \xi_1 \pi' + \cdots + \xi_{e-1}(\pi^e)^{e-1}, \quad \xi_i \in \mathcal{W}_\mathcal{O}(\mathbb{F}).
\]

Since \( \varphi((\pi^i)^{i-1}) = (\varphi(F_1((\pi^i))) = F_1(\varphi((\pi^i))) \), we have

\[
\varphi((\pi^i)^i) = F^{-i-1}(\pi_0(\pi^i)^i + F^{-i-1}(\pi^i)^{i+1} + \cdots + F^{-i-1}(\pi^i)^{e-1+i}),
\]

for all \( i \in \mathbb{Z}_{\geq 0} \). Consider \( R_1 \to R \) as an \( \mathcal{O} \)-pd-thickening by equipping \( aR_1 \) with the trivial \( \mathcal{O} \)-pd-structure. Then the category of \( \mathcal{O} \)-windows over \( \mathcal{W}_{R_1/R} \) is equivalent to the category of \( \mathcal{O} \)-windows over \( \mathcal{W}_R \), hence equivalent to the category of \( \mathcal{O} \)-displays over \( R \) (cf. [3, Proposition 2.21]). Let \( (P^{(1)}, Q^{(1)}, F, F_1) \) be the \( \mathcal{O} \)-window over the frame \( \mathcal{W}_{R_1/R} \) corresponding to \( \mathcal{P} \) via the above equivalence. Then \( P^{(1)} = \mathcal{O}' \otimes \mathcal{W}_\mathcal{O}(R_1) \) and

\[ F_1((\pi^i)^i = (\pi^i)^{i-1} \quad \text{for} \quad i \geq 2, \quad F_1(\pi^i) = \frac{\pi}{\pi^i}. \]

The lifting \( \tilde{\varphi} \in \text{End}(P^{(1)}, Q^{(1)}, F, F_1) \) of \( \varphi \in \text{End}(\mathcal{P}) \) is defined by the same formula (4.4), i.e., we have

\[
\tilde{\varphi} = \varphi \otimes_{\mathcal{W}_\mathcal{O}(\mathbb{F})} \mathcal{W}_\mathcal{O}(R_1).
\]
We need to understand the obstruction to lift \( \tilde{\varphi} \) to a morphism of \( \mathcal{W}_{R_1} \)-windows. The map \( \tilde{\varphi} \) induces an \( \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{W}_1 \)-module homomorphism

\[
Q^{(1)}/I_{\mathcal{O}}(R_1)P^{(1)} \to P^{(1)}/Q^{(1)}.
\]

As an \( R_1 \)-module, \( Q^{(1)}/I_{\mathcal{O}}(R_1)P^{(1)} \) is free with basis \( \{ (\pi')^i - a^i \mid 1 \leq i \leq e - 1 \} \). Here we write \( \pi' \) for \( \pi' \otimes 1 \in \mathcal{O}' \otimes_{\mathcal{O}} R_1 \) and \( a \) for \( 1 \otimes a \). Since \( a^2 = 0 \) in \( R_1 \), it is easy to see that \( (\pi')^i \in Q^{(1)} \) if \( i \geq 2 \). Because \( \tilde{\varphi} \) is an \( \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{W}_1 \)-module homomorphism, \( \tilde{\varphi}(\pi')^i \) is an \( \mathcal{O}' \)-endomorphism of \( \tilde{\varphi}(\pi' - a) \).

Since \( \tilde{\varphi} \) is defined by the formula (4.4), we have

\[
\tilde{\varphi}(\pi - a) = \left( F^{-1} \xi_0(\pi') + F^{-2} \xi_1(\pi')^2 + \cdots + F^{-e} \xi_{e-1}(\pi')^e \right) - \tilde{\varphi}(a)
\]

\[
\equiv F^{-1} \xi_0 \pi' - \xi_0 a \pmod{Q^{(1)}}
\]

\[
\equiv (F^{-1} \xi_0 - \xi_0)a \pmod{Q^{(1)}}.
\]

The lemma follows since \( \text{Lie}(\varphi) = \xi_0 \pmod{\pi} \).

**Proposition 4.3.** With the notation as above. Let \( \mathcal{O}_D = \text{End}(\mathcal{P}) \) be the endomorphism ring, which is isomorphic to the maximal order of the central simple \( E' \)-algebra with invariant \( 1/e \). Let \( \mathcal{O}^u \) be the ring of integers of the maximal unramified extension of \( E' \) with residue field \( \overline{\mathbb{F}} \). Then

\[
\text{End}(\mathcal{P}_{\mathcal{O}^u/(\pi')^{m+1}}) = \mathcal{O}' + (\pi')^m \mathcal{O}_D, \quad m \geq 0.
\]

**Proof.** Let \( \varphi \in \mathcal{O}_D \). Then \( (\pi')^m \varphi \) lifts to an endomorphism of \( \mathcal{P} \) over \( \mathcal{O}^u/(\pi')^m \) by Lemma 4.1. Moreover, we have

\[
\text{Obst}_{m+1}(\pi')^m \varphi = (\pi')^m \text{Obst}_1 \varphi,
\]

where \( (\pi')^m \) on the right hand side denotes the map

\[
(\pi')^m : (a)/(a^2) \otimes_R P/Q \to (a^{(m+1)})/(a^{(m+2)}) P/Q.
\]

Let \( \psi \in (\mathcal{O}' + (\pi')^m \mathcal{O}_D) - (\mathcal{O}' + (\pi')^{m+1} \mathcal{O}_D) \). We claim that \( \psi \) does not lift to an endomorphism of \( \mathcal{P}_{\mathcal{O}^u/(\pi')^{m+2}} \).

Indeed, since \( \pi' \) is a uniformizer of \( \mathcal{O}_D \), we may write

\[
\psi = [a_0] + [a_1] \pi' + \cdots + [a_m] (\pi')^m + \cdots,
\]

where \( a_i \in \mathbb{F}' \). Here \( \mathbb{F}' \) is the degree \( e \) extension of \( \mathbb{F} \). By our assumption on \( \psi \), we have \( a_i \in \mathbb{F} \) for \( i < m \) and \( a_m \notin \mathbb{F} \). Then

\[
\text{Obst}_{m+1} \psi = \text{Obst}_{m+1}([a_m] (\pi')^m + \cdots) = (\pi')^m \text{Obst}_1([a_m] + \pi'[a_{m+1}] + \cdots).
\]

By Lemma 4.2 \( \text{Obst}_{m+1} \psi \) does not vanish since \( \sigma(a_m) \neq a_m \). The claim follows. The proposition then follows from the following lemma.

**Lemma 4.4.** Let \( S \) be an \( \mathcal{O} \)-algebra such that \( \pi \) is nilpotent in \( S \). Let \( \mathfrak{a} \subseteq S \) be an ideal with \( \mathcal{O} \)-pd-structure. Let \( R = S/\mathfrak{a} \). Let \( \mathcal{P} = (P,Q,F,F_1) \) and \( \mathcal{P}' = (P',Q',F,F_1) \) be two \( \mathcal{O} \)-displays over \( S \). Then the natural map

\[
\text{Hom}(\mathcal{P}, \mathcal{P}') \to \text{Hom}(\mathcal{P}_R, \mathcal{P}'_R)
\]

is injective.
Proof. Let \( u : \mathcal{P} \to \mathcal{P}' \) be a morphism of \( \mathcal{O} \)-displays that is zero modulo \( \mathfrak{a} \). Hence \( u(P) \subset W_\mathcal{O}(\mathfrak{a})P \). Since \( S \to R \) is an \( \mathcal{O} \)-pd-thickening, the map \( F_1 : Q' \to P' \) extends to the map \( F_1 : W_\mathcal{O}(\mathfrak{a})P' + Q' \to P' \) which maps \( W_\mathcal{O}(\mathfrak{a})P' \) to \( W_\mathcal{O}(\mathfrak{a})P' \). We claim that the following diagram is commutative

\[
\begin{array}{ccc}
P & \xrightarrow{u} & W_\mathcal{O}(\mathfrak{a})P' \\
\downarrow V^2 & & \downarrow F_1^2 \\
W_\mathcal{O}(S) \otimes_{W_\mathcal{O}(S),F} \mathcal{P} & \xrightarrow{1 \otimes u} & W_\mathcal{O}(S) \otimes_{W_\mathcal{O}(S),F} W_\mathcal{O}(\mathfrak{a})P'.
\end{array}
\]

Here \( F_1^2 \) is the linearization of \( F_1, V^2 : P \to W_\mathcal{O}(S) \otimes_{W_\mathcal{O}(S),F} \mathcal{P} \) is the unique \( W_\mathcal{O}(S) \)-linear map satisfies, for all \( w \in W_\mathcal{O}(S), x \in \mathcal{P} \) and \( y \in Q \), (cf. [2, Lemma 2.2] and [13, Lemma 10])

\[
\begin{align*}
V^2(wFx) &= \pi \cdot w \otimes x, \\
V^2(wF_1y) &= w \otimes y.
\end{align*}
\]

Indeed, since \( P = W_\mathcal{O}(S)\langle F_1Q \rangle \), it suffices to show the commutativity for elements of the form \( wF_1l \), where \( w \in W_\mathcal{O}(S) \) and \( l \in Q \). But in this case the commutativity is obvious, hence the claim holds.

Iterating the diagram, for any \( N \in \mathbb{Z}_{\geq 1} \), we have

\[
(F_1^2)^N(1 \otimes u)(V^2)^N = u.
\]

Therefore \( u = 0 \) since \( \mathcal{P} \) is nilpotent. The lemma follows. \( \square \)

Finally, Theorem [3, Proposition 1.3] follows from Proposition [4, 1.3] and the fact that \( \Gamma_2(\mathcal{O}, \mathcal{O}') \) in [2, Proposition 2.29] is an equivalence on nilpotent objects.

Acknowledgements The author would like to thank Thomas Zink and Hendrik Verhoek for suggestions and comments and thank the support of Grant NSFC 11701272 and Grant 020314803001 of Jiangsu Province (China).

References

[1] T. Ahrendorf, \( \mathcal{O} \)-displays and \( \pi \)-divisible formal \( \mathcal{O} \)-modules. Thesis 2012.