Combinatorics

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Pólya’s Theory of Counting

George Pólya
(1887-1985)
Counting with Symmetry

Rotation:

Rotation & Reflection:
Symmetries
Symmetry

**rotation**

**reflection**

**configuration**

\[\begin{align*}
x : [n] &\rightarrow [m] \\
x^\pi : [n] &\overset{1-1}{\text{on-to}} [n]
\end{align*}\]

**positions**  **colors**

**X** = \([m]^{[n]}\)
Permutation Groups

A group \((G, \cdot)\) with binary operator \(\cdot : G \times G \rightarrow G\)

- **Closure:** \(\pi, \sigma \in G \Rightarrow \pi \cdot \sigma \in G\)
- **Associativity:** \(\pi \cdot (\sigma \cdot \tau) = (\pi \cdot \sigma) \cdot \tau\)
- **Identity:** \(\exists e \in G, \forall \pi \in G, e \cdot \pi = \pi\)
- **Inverse:** \(\forall \pi \in G, \exists \sigma \in G, \pi \cdot \sigma = \sigma \cdot \pi = e\)

A **commutative (abelian) group:** \(\pi \cdot \sigma = \sigma \cdot \pi\)

A **symmetric group** \(S_n\): all permutations

A **cyclic group** \(C_n\): rotations

A **Dihedral group** \(D_n\): rotations & reflections
Permutation Groups

**symmetric group** \( S_n \) : all permutations

\[ \pi : [n] \xrightarrow{1-1 \text{ on-to}} [n] \]

**cyclic group** \( C_n \) : rotations

\[ \pi = (012 \cdots n - 1) \quad \pi(i) = (i + 1) \mod n \]

\( \langle (012 \cdots n - 1) \rangle \) generated by \( (012 \cdots n - 1) \)

**Dihedral group** \( D_n \) : rotations & reflections

\[ \rho(i) = (n - 1) - i \]

generated by \( (012 \cdots n - 1) \) and \( \rho \)
Group Action

Configuration: $x : [n] \rightarrow [m]$  \quad X = [m]^{[n]}$

Permutation: $\pi : [n] \xrightarrow{1-1} [n]$ group $G$

$$(\pi \circ x)(i) = x(\pi(i))$$

Group action: $\circ : G \times X \rightarrow X$

- Associativity: $(\pi \cdot \sigma) \circ x = \pi \circ (\sigma \circ x)$
- Identity: $e \circ x = x$
Graph Isomorphism (GI) Problem

**input:** two undirected graphs $G$ and $H$

**output:** $G \cong H$?

- GI is in NP, but is NOT known to be in P or NPC
- trivial algorithm: $O(n!)$ time
- Babai-Luks ’83: $2^{O(\sqrt{n \log n})}$ time

Babai 2015: a quasi-polynomial time algorithm!

$n^{\text{polylog}(n)} = 2^{\text{polylog}(n)}$
String Isomorphism (SI)

**input:** two strings $x, y : [n] \rightarrow [m]$

a permutation group $G \subseteq S_n$

**output:** $x \cong_G y? \ (\exists \sigma \in G \text{ s.t. } \sigma \circ x = y)$
String Isomorphism (SI)

**input:** two strings \( x, y : [n] \to [m] \)
a permutation group \( G \subseteq S_n \)

**output:** \( x \cong_G y ? \) (\( \exists \sigma \in G \) s.t. \( \sigma \circ x = y \))

A graph \( X(V,E) \) is a string
\[
x : \binom{V}{2} \to \{0,1\} \\
\begin{array}{l}
1: \text{edge} \\
0: \text{no edge}
\end{array}
\]

All permutations on \( V \) induces a permutation group on \( \binom{V}{2} \)

**Johnson group:** \( S_{V}^{(2)} \subseteq S_{\binom{V}{2}} \) on vertex pairs

Two graphs \( X \cong Y \) iff their string versions \( x \cong_{S_{V}^{(2)}} y \)
Orbits

group action \( \circ : G \times X \to X \)

orbit of \( x \): \( Gx = \{ \pi \circ x \mid \pi \in G \} \)

equivalent class of configuration \( x \)

\[ X/G = \{ Gx \mid x \in X \} \]

our goal: count \( |X/G| \)
\[ X = [2]^{[n]} \]

\[ G = C_n \quad \text{and} \quad X/G : \]

\[ G = D_n \quad \text{and} \quad X/G : \]
configuration $x : [n] \rightarrow [m]$

$\vec{v} = (n_1, n_2, \ldots, n_m)$  \text{s.t.}  $n_1 + n_2 + \cdots + n_m = n$

$a_{\vec{v}} : \# \text{ of config. (up to symmetry)}$ with $n_i$ many color $i$

pattern inventory:

(multi-variate) generating function

$$F_G(y_1, y_2, \ldots, y_m) = \sum_{\vec{v}=(n_1,\ldots,n_m)} a_{\vec{v}} y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

with

$$a_{(2,4)} = 3$$
pattern inventory: (multi-variate) generating function

\[ F_G(y_1, y_2, \ldots, y_m) = \sum_{\vec{v} = (n_1, \ldots, n_m)} a_{\vec{v}} y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} \]

\[ a_{\vec{v}} : \# \text{ of config. (up to symmetry) with } n_i \text{ many color } i \]

\[ G = D_n \]

\[ X/G : \]

\[ F_{D_6}(y_1, y_2) = y_1^6 + y_1^5 y_2 + 3y_1^4 y_2^2 + 3y_1^3 y_2^3 + 3y_1^2 y_2^4 + y_1 y_2^5 + y_2^6 \]
\[ \pi = \left( \underbrace{\cdots}_{k \text{ cycles}} \right) \left( \underbrace{\cdots}_{k \text{ cycles}} \right) \left( \underbrace{\cdots}_{k \text{ cycles}} \right) \]

\[ M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i} \]

**cycle index:**

\[ P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n) \]

**pattern inventory:** (multi-variate) generating function

\[ F_G(y_1, y_2, \ldots, y_m) = \sum_{\tilde{\nu}=(n_1, \ldots, n_m)} \prod_{i=1}^{m} a_{\tilde{\nu}} y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} \]

\[ a_{\tilde{\nu}} : \# \text{ of config. (up to symmetry) with } n_i \text{ many color } i \]

**Pólya’s enumeration formula (1937, 1987):**

\[ F_G(y_1, y_2, \ldots, y_m) = P_G \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right) \]
$F_{D_{20}}(r, q, l)$
\[ F_{D20}(r, q, l) = r^{20} + r^{19}q + r^{19}l + 10r^{18}q^2 + 10r^{18}ql + 10r^{18}l^2 + 33r^{17}q^3 + 90r^{17}q^2l + 90r^{17}ql^2 + 33r^{17}l^3 + 145r^{16}q^4 + 489r^{16}q^3l + 774r^{16}q^2l^2 + 489r^{16}ql^3 + 145r^{16}l^4 + 406r^{15}q^5 + 1956r^{15}q^4l + 3912r^{15}q^3l^2 + 3912r^{15}q^2l^3 + 1956r^{15}ql^4 + 406r^{15}l^5 + 1032r^{14}q^6 + 5832r^{14}q^5l + 14724r^{14}q^4l^2 + 19416r^{14}q^3l^3 + 14724r^{14}q^2l^4 + 5832r^{14}ql^5 + 1032r^{14}l^6 + 1980r^{13}q^7 + 3360r^{13}q^6l + 40824r^{13}q^5l^2 + 40824r^{13}q^4l^3 + 67956r^{13}q^3l^4 + 67956r^{13}q^2l^5 + 13608r^{13}ql^6 + 1980r^{13}l^7 + 3260r^{12}q^8 + 25236r^{12}q^7l + 88620r^{12}q^6l^2 + 176484r^{12}q^5l^3 + 221110r^{12}q^4l^4 + 176484r^{12}q^3l^5 + 88620r^{12}q^2l^6 + 25236r^{12}ql^7 + 3260r^{12}l^8 + 4262r^{11}q^9 + 37854r^{11}q^8l + 151416r^{11}q^7l^2 + 352968r^{11}q^6l^3 + 529452r^{11}q^5l^4 + 529452r^{11}q^4l^5 + 352968r^{11}q^3l^6 + 151416r^{11}q^2l^7 + 37854r^{11}ql^8 + 4262r^{11}l^9 + 4752r^{10}l^{10} + 46252r^{10}q^9l + 208512r^{10}q^8l^2 + 554520r^{10}q^7l^3 + 971292r^{10}q^6l^4 + 1164342r^{10}q^5l^5 + 971292r^{10}q^4l^6 + 554520r^{10}q^3l^7 + 208512r^{10}q^2l^8 + 46252r^{10}ql^9 + 4752r^{10}l^{10} + 4262r^9q^{11} + 46252r^9q^{10}l + 231260r^9q^9l^2 + 693150r^9q^8l^3 + 1386300r^9q^7l^4 + 1940568r^9q^6l^5 + 1940568r^9q^5l^6 + 1386300r^9q^4l^7 + 693150r^9q^3l^8 + 231260r^9q^2l^9 + 46252r^9ql^{10} + 4262r^9l^{11} + 3260r^8q^{12} + 37854r^8q^{11}l + 208512r^8q^{10}l^2 + 693150r^8q^9l^3 + 1560534r^8q^8l^4 + 2494836r^8q^7l^5 + 2912112r^8q^6l^6 + 2494836r^8q^5l^7 + 1560534r^8q^4l^8 + 693150r^8q^3l^9 + 208512r^8q^2l^{10} + 37854r^8ql^{11} + 3260r^8l^{12} + 1980r^7q^{13} + 25236r^7q^{12}l + 151416r^7q^{11}l^2 + 554520r^7q^{10}l^3 + 1386300r^7q^9l^4 + 2494836r^7q^8l^5 + 3326448r^7q^7l^6 + 3326448r^7q^6l^7 + 2494836r^7q^5l^8 + 1386300r^7q^4l^9 + 554520r^7q^3l^{10} + 151416r^7q^2l^{11} + 25236r^7ql^{12} + 1980r^7l^{13} + 1032r^6q^{14} + 13608r^6q^{13}l + 88620r^6q^{12}l^2 + 352968r^6q^{11}l^3 + 971292r^6q^{10}l^4 + 1940568r^6q^9l^5
+ 2912112r^6 q^8 l^6 + 3326448r^6 q^7 l^7 + 2912112r^6 q^6 l^8 + 1940568r^6 q^5 l^9 + 971292r^6 q^4 l^{10}
+ 352968r^6 q^3 l^{11} + 88620r^6 q^2 l^{12} + 13608r^6 q l^{13} + 1032r^6 l^{14} + 406r^5 q^{15} + 5832r^5 q^{14} l
+ 40824r^5 q^{13} l^2 + 176484r^5 q^{12} l^3 + 529452r^5 q^{11} l^4 + 1164342r^5 q^{10} l^5 + 1940568r^5 q^{9} l^6
+ 2494836r^5 q^8 l^7 + 2494836r^5 q^7 l^8 + 1940568r^5 q^6 l^9 + 1164342r^5 q^5 l^{10} + 529452r^5 q^4 l^{11}
+ 176484r^5 q^3 l^{12} + 40824r^5 q^2 l^{13} + 5832r^5 q l^{14} + 406r^5 l^{15} + 145r^4 q^{16} + 1956r^4 q^{15} l
+ 14724r^4 q^{14} l^2 + 67956r^4 q^{13} l^3 + 221110r^4 q^{12} l^4 + 529452r^4 q^{11} l^5 + 971292r^4 q^{10} l^6
+ 1386300r^4 q^9 l^7 + 1560534r^4 q^8 l^8 + 1386300r^4 q^7 l^9 + 971292r^4 q^6 l^{10} + 529452r^4 q^5 l^{11}
+ 221110r^4 q^4 l^{12} + 67956r^4 q^3 l^{13} + 14724r^4 q^2 l^{14} + 1956r^4 q l^{15} + 145r^4 l^{16} + 33r^3 q^{17}
+ 489r^3 q^{16} l + 3912r^3 q^{15} l^2 + 19416r^3 q^{14} l^3 + 67956r^3 q^{13} l^4 + 176484r^3 q^{12} l^5
+ 352968r^3 q^{11} l^6 + 554520r^3 q^{10} l^7 + 693150r^3 q^9 l^8 + 693150r^3 q^8 l^9 + 554520r^3 q^7 l^{10}
+ 352968r^3 q^6 l^{11} + 176484r^3 q^5 l^{12} + 67956r^3 q^4 l^{13} + 19416r^3 q^3 l^{14} + 3912r^3 q^2 l^{15}
+ 489r^3 q l^{16} + 33r^3 l^{17} + 10r^2 q^{18} + 90r^2 q^{17} l + 774r^2 q^{16} l^2 + 3912r^2 q^{15} l^3
+ 14724r^2 q^{14} l^4 + 40824r^2 q^{13} l^5 + 88620r^2 q^{12} l^6 + 151416r^2 q^{11} l^7 + 208512r^2 q^{10} l^8
+ 231260r^2 q^9 l^9 + 208512r^2 q^8 l^{10} + 151416r^2 q^7 l^{11} + 88620r^2 q^6 l^{12} + 40824r^2 q^5 l^{13}
+ 14724r^2 q^4 l^{14} + 3912r^2 q^3 l^{15} + 774r^2 q^2 l^{16} + 90r^2 q l^{17} + 10r^2 l^{18} + r q^{19} + 10r q^{18} l
+ 90r q^{17} l^2 + 489r q^{16} l^3 + 1956rq^{15} l^4 + 5832rq^{14} l^5 + 13608rq^{13} l^6 + 25236rq^{12} l^7
+ 37854rq^{11} l^8 + 46252rq^{10} l^9 + 46252rq^9 l^{10} + 37854rq^8 l^{11} + 25236rq^7 l^{12}
+ 13608rq^6 l^{13} + 5832rq^5 l^{14} + 1956rq^4 l^{15} + 489rq^3 l^{16} + 90rq^2 l^{17} + 10rq l^{18} + rl^{19}
+ q^{20} + q^{19} l + 10q^{18} l^2 + 33q^{17} l^3 + 145q^{16} l^4 + 406q^{15} l^5 + 1032q^{14} l^6 + 1980q^{13} l^7
+ 3260q^{12} l^8 + 4262q^{11} l^9 + 4752q^{10} l^{10} + 4262q^9 l^{11} + 3260q^8 l^{12} + 1980q^7 l^{13}
+ 1032q^6 l^{14} + 406q^5 l^{15} + 145q^4 l^{16} + 33q^3 l^{17} + 10q^2 l^{18} + ql^{19} + l^{20}
\[ X = [2]^n \]

\[ G = C_n \quad \quad X/G : \]

\[ G = D_n \quad \quad X/G : \]
Burnside’s Lemma

**group action** \( \circ : G \times X \to X \)

**orbit of** \( x \): \( Gx = \{ \pi \circ x \mid \pi \in G \} \)

\( X/G = \{ Gx \mid x \in X \} \)

**invariant set of** \( \pi \):

\( X_\pi = \{ x \in X \mid \pi \circ x = x \} \)

**Burnside’s Lemma:**

\[ |X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi| \]
group action \( \circ : G \times X \rightarrow X \)

orbit of \( x \): \( Gx = \{ \pi \circ x \mid \pi \in G \} \)

\( X/G = \{ Gx \mid x \in X \} \)

invariant set of \( \pi \): \( X_\pi = \{ x \in X \mid \pi \circ x = x \} \)

stabilizer of \( x \): \( G_x = \{ \pi \in G \mid \pi \circ x = x \} \)

Lemma: \( \forall x \in X, \quad |G_x||Gx| = |G| \)

\[
A(\pi, x) = \begin{cases} 
1 & \pi \circ x = x \\
0 & \text{otherwise}
\end{cases}
\]

\[
|X_\pi| = \sum_{x \in X} A(\pi, x) \quad \quad |G_x| = \sum_{\pi \in G} A(\pi, x)
\]
\[ A(\pi, x) = \begin{cases} 1 & \pi \circ x = x \\ 0 & \text{otherwise} \end{cases} \]

\[ |X_\pi| = \sum_{x \in X} A(\pi, x) \quad |G_x| = \sum_{\pi \in G} A(\pi, x) \]

double counting:

\[ \sum_{\pi \in G} |X_\pi| = \sum_{\pi \in G} \sum_{x \in X} A(\pi, x) = \sum_{x \in X} \sum_{\pi \in G} A(\pi, x) = \sum_{x \in X} |G_x| \]

**Lemma:** \( \forall x \in X, \quad |G_x||Gx| = |G| \)

\[ = |G| \sum_{x \in X} \frac{1}{|Gx|} \]
\[
\sum_{\pi \in G} |X_\pi| = |G| \sum_{x \in X} \frac{1}{|Gx|}
\]

orbits: \(X_1, X_2, \ldots, X_{|X/G|}\)

a partition of \(X\)

\[
= |G| \sum_{i=1}^{X/G} \sum_{x \in X_i} \frac{1}{|Gx|} = |G| \sum_{i=1}^{X/G} \sum_{x \in X_i} \frac{1}{|X_i|}
\]

\[
= |G| \sum_{i=1}^{X/G} 1 = |G||X/G|
\]

\[
|X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi|
\]

Burnside’s Lemma
Burnside’s Lemma

**group action** \( \circ : G \times X \to X \)

**orbit of** \( x \): \( Gx = \{ \pi \circ x \mid \pi \in G \} \)

**invariant set of** \( \pi \):

\[
X_\pi = \{ x \in X \mid \pi \circ x = x \}
\]

**Burnside’s Lemma:**

\[
|X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi|
\]
Cycle Decomposition

permutation \( \pi : [n] \xrightarrow{\text{1-1}} [n] \)

\[
\begin{pmatrix}
0 & 1 & 2 & \cdots & n-1 \\
\pi(0) & \pi(1) & \pi(2) & \cdots & \pi(n-1)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
2 & 4 & 0 & 1 & 3
\end{pmatrix} = (0 \ 2)(1 \ 4 \ 3)
\]
Cycle Decomposition

**permutation**

\[ \pi : [n] \xrightarrow{1-1} [n] \]

- Every cycle of \( \pi \) has the same color

\[ \forall i \in [n], \ x(\pi(i)) = x(i) \]
every cycle of $\pi$ has the same color

\[ \forall i \in [n], \ x(\pi(i)) = x(i) \]

\[ X = [m]^n \quad m\text{-colorings of } n \text{ positions} \]

\[ \pi = (\cdots)(\cdots) \cdots (\cdots) \]

\[ |X_\pi| = |\{ x \in X \mid \pi \circ x = x \}| = m^k \]

Burnside's Lemma:

\[ |X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi| = \frac{1}{|G|} \sum_{\pi \in G} m^{\#\text{cycle}(\pi)} \]
\[X = [2]^{[n]}\]

\[G = C_n\]

\[X/G:\]

\[G = D_n\]

\[X/G:\]
configuration \( x : [n] \to [m] \)

\[
\begin{array}{c}
\text{configuration} \\
\includegraphics[width=0.5\textwidth]{configuration.png} \\
\end{array}
\]

\[ a_{(2,4)} = 3 \]

\( \vec{v} = (n_1, n_2, \ldots, n_m) \quad \text{s.t.} \quad n_1 + n_2 + \cdots + n_m = n \)

\( a_{\vec{v}} : \# \text{ of config. (up to symmetry) with } n_i \text{ many color } i \)

pattern inventory :

(multi-variate) generating function

\[
F_G(y_1, y_2, \ldots, y_m) = \sum_{\vec{v}=(n_1,\ldots,n_m)} a_{\vec{v}} y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}
\]

\( \quad n_1 + \cdots + n_m = n \)
pattern inventory: (multi-variate) generating function

\[ F_G(y_1, y_2, \ldots, y_m) = \sum_{\vec{v}=(n_1, \ldots, n_m)} a_{\vec{v}} \ y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} \]

\[ n_1 + \cdots + n_m = n \]

\[ a_{\vec{v}} : \# \ of \ config. \ (up \ to \ symmetry) \ with \ n_i \ many \ color \ i \]

\[ G = D_n \]

\[ X/G : \]

\[ F_{D_6}(y_1, y_2) = y_1^6 + y_1^5 y_2 + 3y_1^4 y_2^2 + 3y_1^3 y_2^3 + 3y_1^2 y_2^4 + y_1 y_2^5 + y_2^6 \]
Cycle index

permutation \( \pi : [n] \xrightarrow{1-1 \text{ on-to}} [n] \) from group \( G \)

\[
\pi = \underbrace{(...)}_{\ell_1} \underbrace{(...)}_{\ell_2} \cdots \underbrace{(...)}_{\ell_k}
\]

k cycles

monomial: \( M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x^{\ell_i} \)

cycle index:

\[
P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n)
\]

Burnside’s Lemma:

\[
|X/G| = P_G(m, m, \ldots, m) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(m, m, \ldots, m)
\]
\[ \pi = (\cdots)(\cdots) \ldots (\cdots) \text{ k cycles} \]

\[ M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i} \]

cycle index: \[ P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n) \]

pattern inventory: (multi-variate) generating function

\[ F_G(y_1, y_2, \ldots, y_m) = \sum_{\bar{v}=(n_1, \ldots, n_m) \atop n_1 + \cdots + n_m = n} a_{\bar{v}} \ y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} \]

\[ a_{\bar{v}} : \# \text{ of config. (up to symmetry) with } n_i \text{ many color } i \]

\[ Pólya's \ enumeration \ formula \ (1937, 1987): \]

\[ F_G(y_1, y_2, \ldots, y_m) = P_G \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right) \]

\[ \vec{v} = (n_1, n_2, \ldots, n_m) \quad n_1 + n_2 + \cdots + n_m = n \]

\[ X = [m]^n \quad X^{\vec{v}} = \{ x \in [m]^n \mid \forall i \in [m], x^{-1}(i) = n_i \} \]

**Invariant set of \( \pi \):** \[ X^{\vec{v}}_\pi = \{ x \in X^{\vec{v}} \mid \pi \circ x = x \} \]

\[ X^{(2,4)} : \]

\[ a_{(2,4)} = 3 \]

**Burnside’s Lemma:**

\[ a_{\vec{v}} = \frac{1}{|G|} \sum_{\pi \in G} |X^{\vec{v}}_\pi| \]
\[ \pi = \underbrace{(\cdots)(\cdots) \cdots (\cdots)}_{k \text{ cycles}} \]

\[ X_{\pi}^{\tilde{v}} = \{ x \in X^{\tilde{v}} \mid \pi \circ x = x \} \]

\[ \forall i \in [n], \ x(\pi(i)) = x(i) \]

\[ M_{\pi} \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right) = \]

\[ (y_1^{\ell_1} + y_2^{\ell_1} + \cdots + y_m^{\ell_1})(y_1^{\ell_2} + y_2^{\ell_2} + \cdots + y_m^{\ell_2}) \cdots (y_1^{\ell_k} + y_2^{\ell_k} + \cdots + y_m^{\ell_k}) \]

\[ = \sum \left| X_{\pi}^{\tilde{v}} \right| y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} \]

\[ \text{recall: } M_{\pi}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i} \]
\[
X_{\pi}^{\vec{v}} = \{ x \in X^{\vec{v}} \mid \pi \circ x = x \}
\]

\[
M_{\pi} \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right) = \sum_{\vec{v} = (n_1, \ldots, n_m) \atop n_1 + \cdots + n_m = n} |X_{\pi}^{\vec{v}}| y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}
\]

Burnside's Lemma:
\[
a_{\vec{v}} = \frac{1}{|G|} \sum_{\pi \in G} |X_{\pi}^{\vec{v}}|
\]

pattern inventory:
\[
F_G(y_1, y_2, \ldots, y_m)
\]
\[
= \sum_{\vec{v} = (n_1, \ldots, n_m) \atop n_1 + \cdots + n_m = n} a_{\vec{v}} \ y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} = \sum_{\vec{v} = (n_1, \ldots, n_m) \atop n_1 + \cdots + n_m = n} \left( \frac{1}{|G|} \sum_{\pi \in G} |X_{\pi}^{\vec{v}}| \right) y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}
\]

\[
= \frac{1}{|G|} \sum_{\pi \in G} \sum_{\vec{v} = (n_1, \ldots, n_m) \atop n_1 + \cdots + n_m = n} |X_{\pi}^{\vec{v}}| y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}
\]

cycle index:
\[
= \frac{1}{|G|} \sum_{\pi \in G} M_{\pi} \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right) = P_G \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right)
\]
\[
\pi = \underbrace{\cdots\cdots\cdots\cdots}_{k \text{ cycles}}
\]

\[M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i}\]

cycle index: \[P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n)\]

pattern inventory: (multi-variate) generating function

\[F_G(y_1, y_2, \ldots, y_m) = \sum_{\vec{v} = (n_1, \ldots, n_m) \atop n_1 + \cdots + n_m = n} a_{\vec{v}} y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}\]

\[a_{\vec{v}}: \# \text{ of config. (up to symmetry) with } n_i \text{ many color } i\]

Pólya's enumeration formula (1937,1987):

\[F_G(y_1, y_2, \ldots, y_m) = P_G \left( \sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n \right)\]
\[ F_{D_{20}}(r, q, l) = r^{20} + r^{19}q + r^{19}l + 10r^{18}q^2 + 10r^{18}ql + 10r^{18}l^2 + 33r^{17}q^3 + 90r^{17}ql + 90r^{17}l^2 + 33r^{17}l^3 + 145r^{16}q^4 + 489r^{16}q^2l + 774r^{16}ql^2 + 489r^{16}l^3 + 145r^{16}l^4 + 406r^{15}q^5 + 1956r^{15}q^3l + 3912r^{15}q^2l^2 + 3912r^{15}ql^3 + 1956r^{15}l^4 + 406r^{15}l^5 + 1032r^{14}q^6 + 5832r^{14}q^4l + 14724r^{14}q^3l^2 + 19416r^{14}q^2l^3 + 14724r^{14}ql^4 + 5832r^{14}l^5 + 1032r^{14}l^6 + 1980r^{13}q^7 + 13608r^{13}q^5l + 40824r^{13}q^4l^2 + 67956r^{13}q^3l^3 + 40824r^{13}q^2l^4 + 13608r^{13}ql^5 + 1980r^{13}l^6 + 3260r^{12}q^8 + 25236r^{12}q^6l + 88620r^{12}q^5l^2 + 17648r^{12}q^4l^3 + 221110r^{12}q^3l^4 + 17648r^{12}q^2l^5 + 88620r^{12}ql^6 + 25236r^{12}l^7 + 3260r^{12}l^8 + 4262r^{11}q^9 + 37854r^{11}q^7l + 151416r^{11}q^5l^2 + 352956r^{11}q^4l^3 + 529452r^{11}q^3l^4 + 529452r^{11}ql^5 + 352956r^{11}l^6 + 151416r^{11}l^7 + 37854r^{11}l^8 + 4262r^{11}l^9 + 4752r^{10}q^{10} + 46252r^{10}q^8l + 208512r^{10}q^7l^2 + 554520r^{10}q^6l^3 + 971292r^{10}q^5l^4 + 1164342r^{10}q^4l^5 + 971292r^{10}q^3l^6 + 208512r^{10}q^2l^7 + 46252r^{10}ql^8 + 4752r^{10}l^9 + 4262r^9q^{11} + 46252r^9q^9l + 231260r^9q^8l^2 + 693150r^9q^7l^3 + 1386300r^9q^6l^4 + 9180q^7 + 253236r^9q^5l^5 + 1386300r^9q^4l^6 + 9180q^7 + 231260r^9q^3l^7 + 693150r^9q^2l^8 + 1386300r^9ql^9 + 9180q^7 + 231260r^9l^{10} + 693150r^9l^{11} + 1386300r^9l^{12} + 9180q^7 + 231260 \]
\[ F_{D20}(r, q, l) = r^{20} + r^{19} q + r^{19} l + 10r^{18} q^2 + 10r^{18} ql + 10r^{18} l^2 + 33r^{17} q^3 + 90r^{17} q^2 l + 90r^{17} ql^2 + 33r^{17} l^3 + 145r^{16} q^4 + 489r^{16} q^3 l + 774r^{16} q^2 l^2 + 489r^{16} q^3 l^3 + 145r^{16} l^4 + 406r^{15} q^5 + 1956r^{15} q^4 l + 3912r^{15} q^3 l^2 + 3912r^{15} q^2 l^3 + 1956r^{15} q^4 l^4 + 406r^{15} l^5 + 1032r^{14} q^6 + 5832r^{14} q^5 l + 14724r^{14} q^4 l^2 + 19416r^{14} q^3 l^3 + 14724r^{14} q^2 l^4 + 5832r^{14} q^5 l^5 + 1032r^{14} q^6 l^6 + 1980r^{13} q^7 + 13608r^{13} q^6 l + 40824r^{13} q^5 l^2 + 67956r^{13} q^4 l^3 + 67956r^{13} q^3 l^4 + 40824r^{13} q^2 l^5 + 13608r^{13} q^6 l + 40824r^{13} q^5 l^2 + 3260r^{12} q^8 + 25236r^{12} q^7 l + 88620r^{12} q^6 l^2 + 176484r^{12} q^5 l^3 + 221110r^{12} q^4 l^4 + 176484r^{12} q^3 l^5 + 88620r^{12} q^2 l^6 + 25236r^{12} q^7 l^7 + 3260r^{12} l^8 + 4262r^{11} q^9 + 37854r^{11} q^8 l + 151416r^{11} q^7 l^2 + 352968r^{11} q^6 l^3 + 529452r^{11} q^5 l^4 + 529452r^{11} q^4 l^5 + 352968r^{11} q^3 l^6 + 151416r^{11} q^2 l^7 + 37854r^{11} q^8 l^8 + 4262r^{11} l^9 + 4752r^{10} q^{10} + 46252r^{10} q^9 l + 208512r^{10} q^8 l^2 + 554520r^{10} q^7 l^3 + 971292r^{10} q^6 l^4 + 1164342r^{10} q^5 l^5 + 971292r^{10} q^4 l^6 + 554520r^{10} q^3 l^7 + 208512r^{10} q^2 l^8 + 46252r^{10} q^9 l^9 + 4752r^{10} l^{10} + 4262r^9 q^{11} + 46252r^9 q^{10} l + 231260r^9 q^9 l^2 + 693150r^9 q^8 l^3 + 1386300r^9 q^7 l^4 + 1940568r^9 q^6 l^5 + 1940568r^9 q^5 l^6 + 1386300r^9 q^4 l^7 + 693150r^9 q^3 l^8 + 231260r^9 q^2 l^9 + 46252r^9 q^10 l + 4262r^9 q^{11} + 3260r^8 q^{12} + 37854r^8 q^{11} l + 208512r^8 q^{10} l^2 + 693150r^8 q^9 l^3 + 1560534r^8 q^8 l^4 + 2494836r^8 q^7 l^5 + 2912112r^8 q^6 l^6 + 2494836r^8 q^5 l^7 + 1560534r^8 q^4 l^8 + 693150r^8 q^3 l^9 + 208512r^8 q^2 l^{10} + 37854r^8 q^{11} l + 3260r^8 l^{12} + 1980r^7 q^{13} + 25236r^7 q^{12} l + 151416r^7 q^{11} l^2 + 554520r^7 q^{10} l^3 + 1386300r^7 q^9 l^4 + 2494836r^7 q^8 l^5 + 3326448r^7 q^7 l^6 + 3326448r^7 q^6 l^7 + 2494836r^7 q^5 l^8 + 1386300r^7 q^4 l^9 + 554520r^7 q^3 l^{10} + 151416r^7 q^2 l^{11} + 25236r^7 q^{12} l^2 + 1980r^7 l^{13} + 1032r^6 q^{14} + 13608r^6 q^{13} l + 88620r^6 q^{12} l^2 + 352968r^6 q^{11} l^3 + 971292r^6 q^{10} l^4 + 1940568r^6 q^9 l^5 + \ldots \]
Orbits of Group Actions