Combinatorics

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Matching Theory
System of Distinct Representatives
(Transversal)

system of distinct representatives (SDR)

for sets \( S_1, S_2, \ldots, S_m \)

distinct \( x_1, x_2, \ldots, x_m \)

\( x_i \in S_i \)

for \( i = 1, 2, \ldots, m \)
Marriage Problem

Does there exist an SDR for $S_1, S_2, \ldots, S_m$?

$m$ girls

$S_i :$ boys that girl $i$ likes

“Is there a way of marrying these $m$ girls?”
$S_1, S_2, \ldots, S_m$ have a SDR

$\exists$ distinct $x_1 \in S_1, x_2 \in S_2, \ldots, x_m \in S_m$

$\forall I \subseteq \{1, 2, \ldots, m\}$,

$$|\bigcup_{i \in I} S_i| \geq |\{x_i \mid i \in I\}| \geq |I|.$$
$S_1, S_2, \ldots, S_m$ have a SDR $\forall I \subseteq \{1, 2, \ldots, m\}, \quad |\bigcup_{i \in I} S_i| \geq |I|.$
Hall’s Theorem (marriage theorem)

$S_1, S_2, \ldots, S_m$ have a SDR if and only if

\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad |\bigcup_{i \in I} S_i| \geq |I|. \]
Hall’s Theorem (graph theory form)
A bipartite graph \( G(U, V, E) \) has a matching of \( U \)
\[ |N(S)| \geq |S| \text{ for all } S \subseteq U \]

matching: edge independent set
\( M \subseteq E \) with
\( \text{no } e_1, e_2 \in M \text{ share a vertex} \)
\[ N(S) = \{ v \mid \exists u \in S, uv \in E \} \]
Hall’s Theorem (marriage theorem)
\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad \left| \bigcup_{i \in I} S_i \right| \geq |I|. \]
\[ \Rightarrow \quad S_1, S_2, \ldots, S_m \text{ have a SDR} \]

**critical family:** \( S_1, S_2, \ldots, S_k \quad k < m \)

\[
\left| \bigcup_{i=1}^{k} S_i \right| = k
\]

**Induction on** \( m \): \( m = 1, \) trivial

**case. 1:** there is no critical family in \( S_1, S_2, \ldots, S_m \)

**case. 2:** there is a critical family in \( S_1, S_2, \ldots, S_m \)
case 1: there is no critical family in $S_1, S_2, \ldots, S_m$

$\forall I \subseteq \{1, 2, \ldots, m\}$ that $|I| < m,$ $|\bigcup_{i \in I} S_i| > |I|$

take an arbitrary $x \in S_m$ as representative of $S_m$

remove $S_m$ and $x$ $S_i' = S_i \setminus \{x\}$ $i = 1, 2, \ldots, m-1$

$\forall I \subseteq \{1, 2, \ldots, m-1\},$ $|\bigcup_{i \in I} S_i'| \geq |I|$

due to I.H. $S_1', \ldots, S_{m-1}'$ have a SDR $\{x_1, \ldots, x_{m-1}\}$

$x_1, \ldots, x_{m-1}$ and $x$ form a SDR for $S_1, S_2, \ldots, S_m$
Hall's Theorem (marriage theorem)
\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad \left| \bigcup_{i \in I} S_i \right| \geq |I|. \]

\[ S_1, S_2, \ldots, S_m \text{ have a SDR} \]

**case.2:** there is a critical family in \( S_1, S_2, \ldots, S_m \)
say \[ |S_{m-k+1} \cup \cdots \cup S_m| = k \quad k < m \]
due to I.H. \( S_{m-k+1}, \ldots, S_m \) have a SDR \( X = \{x_1, \ldots, x_k\} \)

\[ S_i' = S_i \setminus X \quad i = 1, 2, \ldots, m-k \]

\[ \forall I \subseteq \{1, 2, \ldots, m-k\}, \quad \left| \bigcup_{i=m-k+1}^{m} S_i \cup \bigcup_{i \in I} S_i \right| \geq k + |I| \]

\[ \Rightarrow \left| \bigcup_{i \in I} S'_i \right| \geq |I| \]
Hall’s Theorem (marriage theorem)
\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad |\bigcup_{i \in I} S_i| \geq |I|. \]

\[ S_1, S_2, \ldots, S_m \text{ have a SDR} \]

case.2: there is a critical family in \( S_1, S_2, \ldots, S_m \)
say \[ |S_{m-k+1} \cup \cdots \cup S_m| = k \quad k < m \]
due to l.H. \( S_{m-k+1}, \ldots, S_m \) have a SDR \( X = \{x_1, \ldots, x_k\} \)
\[ S_i' = S_i \setminus X \quad i = 1, 2, \ldots, m-k \]
\[ \forall I \subseteq \{1, 2, \ldots, m-k\}, \quad \bigcup_{i \in I} S_i' \geq |I| \]
due to l.H.
\[ S_1', \ldots, S_{m-k}' \text{ have a SDR} \quad Y = \{y_1, \ldots, y_{m-k}\} \]
\[ X \text{ and } Y \text{ form a SDR for } S_1, S_2, \ldots, S_m \]
Hall’s Theorem (marriage theorem)

$S_1, S_2, \ldots, S_m$ have a SDR if and only if

\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad \left| \bigcup_{i \in I} S_i \right| \geq |I|. \]
Min-Max Theorems

• König-Egerváry theorem: in bipartite graph
  \( \min \) vertex cover = \( \max \) matching

• Menger’s theorem: in graph
  \( \min \) vertex-cut = \( \max \) vertex-disjoint paths

• Dilworth’s theorem: in poset
  \( \min \) chain-decomposition = \( \max \) antichain
König-Egerváry theorem

**Theorem** (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

**Matching:** \( M \subseteq E \)

- no \( e_1, e_2 \in M \) share a vertex

**Vertex Cover:** \( C \subseteq V \)

- all \( e \in E \) adjacent to some \( v \in C \)
**Theorem** (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

Matching:

- Independent 1s
- Do not share row/column
Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.
**Theorem** (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

**König-Egerváry Theorem** *(matrix form)*

For any 0-1 matrix, the maximum number of independent 1's equals the minimum number of rows and columns required to cover all the 1's.
$A$: $m \times n$ 0-1 matrix
$r$: max # of independent 1’s
$s$: min # of rows/columns covering all 1’s

any $r$ independent 1’s requires $r$ rows/columns to cover
$A: m \times n$ 0-1 matrix

$r: \text{max \# of independent 1's}$

$s: \text{min \# of rows/columns covering all 1's}$

$r \geq s$

min covering: $s = a \text{ rows} + b \text{ columns}$

$A = \begin{bmatrix}
B_{a \times b} & C_{a \times (n-b)} \\
D_{(m-a) \times b} & 0
\end{bmatrix}$

$C \text{ has } a \text{ independent 1's}$

$D \text{ has } b \text{ independent 1's}$
$A$ has min covering: $s = a$ rows + $b$ columns

$A = \begin{bmatrix} B_{a \times b} & C_{a \times (n-b)} \\ D_{(m-a) \times b} & 0 \end{bmatrix}$

$C$ has $a$ independent 1’s

$S_i = \{ j \mid C_{ij} = 1 \}$

$S_1, S_2, \ldots, S_a$ have a SDR

otherwise $\exists 1 \leq |I| \leq a, \quad |\bigcup_{i \in I} S_i| < |I|$  \hfill (Hall)

$C$ can be covered by $(a - |I|)$ rows + $|\bigcup_{i \in I} S_i|$ columns
A has min covering: $s = a$ rows + $b$ columns

$$A = \begin{bmatrix} B_{a \times b} & C_{a \times (n-b)} \\ D_{(m-a) \times b} & 0 \end{bmatrix}$$

C has $a$ independent 1’s

$$S_i = \{ j \mid C_{ij} = 1 \}$$

$S_1, S_2, \ldots, S_a$ have a SDR

otherwise $\exists 1 \leq |I| \leq a$, $|\bigcup_{i \in I} S_i| < |I|$ (Hall)

C can be covered by $< a$ rows&columns

A can be covered by $< a + b$ rows&columns

contradiction!
A has min covering: \( s = a \) rows + \( b \) columns

\[
A = \begin{bmatrix}
B_{a \times b} & C_{a \times (n-b)} \\
D_{(m-a) \times b} & 0
\end{bmatrix}
\]

\( C \) has \( a \) independent 1’s

\[
S_i = \{ j \mid C_{i,j} = 1 \}
\]

\( S_1, S_2, \ldots, S_a \) have a SDR

SDR: distinct \( j_1, j_2, \ldots, j_a \)

\( C(i, j_i) = 1 \)
$A : m \times n$ 0-1 matrix

$r : \text{max } \# \text{ of independent 1's}$

$s : \text{min } \# \text{ of rows/columns covering all 1's}$

$r \geq s$

$A$ has min covering: $s = a \text{ rows } + b \text{ columns}$

$$A = \begin{bmatrix}
B_{a \times b} & C_{a \times (n-b)} \\
D_{(m-a) \times b} & 0
\end{bmatrix}$$

$C$ has $a$ independent 1's

$D$ has $b$ independent 1's
König-Egerváry Theorem (matrix form)

For any 0-1 matrix, the maximum number of independent 1's equals the minimum number of rows and columns required to cover all the 1's.

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.
Chains and Antichains
\( \mathcal{F} \subseteq 2^{[n]} \) with \( \subseteq \) define a partially ordered set (poset)

reflexivity: \( A \subseteq A \)

antisymmetry:
\( A \subseteq B \) and \( B \subseteq A \) \( \Rightarrow \) \( A = B \)

transitivity:
\( A \subseteq B \) and \( B \subseteq C \) \( \Rightarrow \) \( A \subseteq C \)

chain:
\( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r \)

antichain:
\( A_1, A_2, \ldots, A_r \) that \( \forall A_i, A_j, \ A_i \nsubseteq A_j \)
Dilworth’s Theorem

Size of the largest antichain in the poset $P = \text{size of the smallest partition of } P \text{ into chains.}$
Dilworth’s Theorem

Size of the largest antichain in the poset $P = \text{size of the smallest partition of } P \text{ into chains.}$

Suppose: $P$ has an antichain of size $r$.

$P$ can be partitioned to $s$ chains.

$$r \leq s$$

antichain $A$, chain $C$ \quad |A \cap C| \leq 1

We only need to prove:

There exist an antichain $A \subseteq P$ of size $r$ and a partition of $P$ into $r$ chains.
König-Egerváry Theorem:

\[ \exists \text{ matching } M \text{ and vertex cover } C, \quad |M| = |C| = k \]

\( x \in P \) uncovered by \( C \) \quad \Rightarrow \quad \text{antichain} \quad \geq n-k

otherwise \quad \text{\( C \) is not a vertex cover}
poset $P$

$G(U,V,E)$

$U = V = P$

$uv \in E$ if $u < v$

$\exists$ matching $M$ and vertex cover $C$, $|M| = |C| = k$

$\exists$ antichain of size $\geq n-k$

decompose $P$ into chains:

$u, v$ in the same chain if $uv \in M$

$\#$ chains $= \#$ unmatched vertices in $U = n-k$
Dilworth’s Theorem

Suppose that the largest antichain in the poset $P$ has size $r$. Then $P$ can be partitioned into $r$ chains.

There exists an antichain $A \subseteq P$ and a partition of $P$ into $r$ chains such that $|A| = r$. 

$\exists$ antichain of size $\geq n-k = \# \text{ chains}$
Hall's Theorem (marriage theorem)
\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad |\bigcup_{i \in I} S_i| \geq |I|. \]

\[ S_1, S_2, \ldots, S_m \text{ have a SDR} \]

let \( X = S_1 \cup \cdots \cup S_m \)

poset \( P: \quad X \cup \{S_1, \ldots, S_m\} \)

\[ x < S_i \quad \text{if} \quad x \in S_i \]

\( X \) is the largest antichain in \( P \).

\( A \subseteq P \) is an antichain \quad \( I = \{i \mid S_i \in A\} \quad S_I = \bigcup_{i \in I} S_i \)

\[ A \cap S_I = \emptyset \quad \rightarrow \quad |A| \leq |I| + |X| - |S_I| \leq |X| \]

Hall condition
Hall’s Theorem (marriage theorem)
\[ \forall I \subseteq \{1, 2, \ldots, m\}, \quad \left| \bigcup_{i \in I} S_i \right| \geq |I|. \]

\[ S_1, S_2, \ldots, S_m \text{ have a SDR} \]

let \( X = S_1 \cup \cdots \cup S_m \)

poset \( P: \quad X \cup \{S_1, \ldots, S_m\} \)

\[ x < S_i \text{ if } x \in S_i \]

\( X \) is the largest antichain in \( P \).

Dilworth: \( P \) is partitioned into \( n=|X| \) chains
\[ \{S_1, x_1\}, \{S_2, x_2\}, \ldots, \{S_m, x_m\}, \{x_{m+1}\}, \ldots, \{x_n\} \]
Erdős-Szekeres Theorem

A sequence of \( > mn \) different numbers must contain either an increasing subsequence of length \( m+1 \), or a decreasing subsequence of length \( n+1 \).

\((a_1, \ldots, a_N)\) of \( N \) different numbers \( N > mn \)

poset \( P: \{ (i, a_i) \mid i = 1, 2, \ldots, N \} \)

\((i, a_i) \leq (j, a_j) \) if \( a_i \leq a_j \) and \( i \leq j \)

chain: increasing subseq
antichain: decreasing subseq

Use Dilworth!
Birkhoff - von Neumann Theorem
Every doubly stochastic matrix is a convex combination of permutation matrix.

doubly stochastic matrix $A$: $n \times n$ $A_{ij} \geq 0$

\[ \forall j, \sum_i A_{ij} = 1 \quad \text{and} \quad \forall i, \sum_j A_{ij} = 1 \]

permutation matrix $P$: $P_{ij} \in \{0, 1\}$
every row/column has precisely one 1

convex combination:

\[ A = \sum_{i=1}^{m} \lambda_i P_i \quad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = 1 \]
\( n \times n \) nonnegative matrix \( A \):

\[
\forall j, \sum_{i} A_{ij} = \gamma \quad \forall i, \sum_{j} A_{ij} = \gamma \quad \gamma > 0
\]

\[
A = \sum_{i=1}^{m} \lambda_i P_i \quad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = \gamma
\]

induction on \# of non-zeros in \( A \) \hspace{1cm} \text{denoted } m

\( \gamma > 0 \quad \Rightarrow \quad m \geq n \quad \text{Basis: } m=n \)

\( S_i = \{ j \mid A_{ij} > 0 \} \quad i = 1, 2, \ldots, n \)

If \( \exists I \subseteq \{1, \ldots, n\}, \left| \bigcup_{i \in I} S_i \right| < |I| \)

\[
< |I| \quad \text{sum by columns} < |I| \gamma \quad \text{contradiction!}
\]

\[
\sum_{i \in I} = |I| \gamma \quad \text{sum by rows}
\]
\( n \times n \) nonnegative matrix \( A \):

\[
\forall j, \sum_{i} A_{ij} = \gamma \quad \forall i, \sum_{j} A_{ij} = \gamma \quad \gamma > 0
\]

\[
A = \sum_{i=1}^{m} \lambda_i P_i \quad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = \gamma
\]

induction on \# of non-zeros in \( A \) denoted \( m \)

\[
S_i = \{ j \mid A_{ij} > 0 \} \quad i = 1, 2, \ldots, n
\]

\[
\forall I \subseteq \{ 1, \ldots, n \}, \left| \bigcup_{i \in I} S_i \right| \geq |I|
\]

Hall’s Thm: \( \exists \) SDR \( j_1 \in S_1, \ldots, j_n \in S_n \)

permutation matrix \( P_m(i, j_i) = 1 \) otherwise \( = 0 \)

\[
\lambda_m = \min_{1 \leq i \leq n} A(i, j_i) \quad A' = A - \lambda_m P_m
\]

\[
\gamma' = \gamma - \lambda_m \quad m' \leq m - 1
\]
Birkhoff - von Neumann Theorem

Every doubly stochastic matrix is a convex combination of permutation matrix.

doubly stochastic matrix $A$: $n \times n$ $A_{ij} \geq 0$

$$\forall j, \sum_i A_{ij} = 1 \quad \text{and} \quad \forall i, \sum_j A_{ij} = 1$$

permutation matrix $P$: $P_{ij} \in \{0, 1\}$

every row/column has precisely one 1

convex combination:

$$A = \sum_{i=1}^{m} \lambda_i P_i \quad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = 1$$