The tame kernels of number fields

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Abstract. We review some progress on the tame kernels of number fields, especially on densities for certain sets of primes for which the \( p^n \)-rank of the tame kernel of certain related number fields has some fixed value.

1. Introduction

Let \( R \) be a ring. The definition of \( K_2 R \) was given by Milnor in 1967. For a field \( F \), Matsumoto proved in 1969 that \( K_2 F \) is the quotient of \( F^\times \otimes F^\times \) by the subgroup generated by the elements \( a \otimes (1 - a) \), where \( a \neq 0, 1 \). One can see [31] for details of a proof of Matsumoto’s Theorem.

Let \( F \) be a number field, \( \mathcal{O}_F \) the ring of integers of \( F \), \( r_1 \) the number of real places of \( F \), \( r_2 \) the number of complex places of \( F \). Let \( \text{Cl}(F) \) be the class group of \( \mathcal{O}_F \), and let \( \text{Cl}_2(F) \) be the subgroup of \( \text{Cl}(F) \) generated by classes containing prime ideals dividing 2. Bass proved in [1] and [2] that the following complex

\[
K_2 \mathcal{O}_F \longrightarrow K_2 F \xrightarrow{(\partial_p)} \bigoplus_p k_p^\times,
\]

is an exact sequence, where \( p \) runs through all finite primes of \( \mathcal{O}_F \) and \( k_p \) is the residue field \( \mathcal{O}_F / p \) and \( (\partial_p) \) is the tame mapping. One can see Lemma 11.5 of [31] for the definition of the tame mapping. Hence \( K_2 \mathcal{O}_F \) is also called the tame kernel. Garland proved in [17] that \( K_2 \mathcal{O}_E \) is a finite abelian group. In [12], Coates related \( K_2 \) to some classical invariants of number fields. In 1976, Tate proved in [51] that if \( F \) contains a primitive \( l \)-th root of unity \( z \) and \( \Delta \) is the group of elements \( a \in F^\times \) such that \( \{ z, a \} = 1 \), then every element of \( iK_2 F \) is of form \( \{ z, b \} \) for some \( b \in F^\times \). Moreover, we have \( (\Delta : (F^\times)^l) = l^{r_2 + 1} \). If \( F \) contains a primitive \( l \)-th root of unity, one can deduce the \( p \)-rank formula of \( K_2 \mathcal{O}_F \) from Tate’s theorem (cf. Corollary 3.9 of [27]).

In case of quadratic number fields, Browkin and Schinzel gave in [4] the explicit expression of the 2-torsion elements of \( K_2 \mathcal{O}_F \). In [8], one can find a table of tame and wild kernels of quadratic imaginary number fields whose discriminant \( D > -5000 \).
For any finite abelian group $G$, let $r_{2^n}$ be the $2^n$-rank of $G$, i.e.,

$$r_{2^n} = \log_2 |G^{2^n-1}/G^{2^n}|.$$ 

In 1992, Qin determined explicitly the structure of the 2-Sylow subgroup of $K_2 \mathcal{O}_F$ for some real quadratic number fields in his Ph.D. thesis. By this method, the 4-rank of $K_2 \mathcal{O}_F$ can be obtained by considering the Legendre symbols. Qin gave an efficient way to compute the 4-rank of $K_2 \mathcal{O}_F$ for arbitrary quadratic number fields in [39] and [40]. One can also find from [39] and [40] the explicit tables of the 4-ranks of tame kernels of $K_2 \mathcal{O}_F$, where the number of odd prime factors of $d$ is less than or equal to 3. Comparing with the classical results of [46] and [47] on the 4-rank of class group of $\mathcal{O}_F$, one can find that the results in [39] and [40] in fact relates the 4-rank of $K_2 \mathcal{O}_F$ with the 4-rank of class group of $\mathcal{O}_F$. Since the density of 4-rank of class group of $\mathcal{O}_F$ is already known in [15] and [14], we can get the density of the 4-rank of $K_2 \mathcal{O}_F$ by this method. If the number of prime factors of the discriminant is small, the density of 4-ranks can be found in [35], [36], [37] and [10]. Hurrelbrink and Kolster introduced in [26] a kind of sign matrices to compute $r_4(K_2 \mathcal{O}_F)$ for relative quadratic extensions, which is via the local Hilbert symbols. One can also see [13] for a similar sign matrix to compute $r_4(K_2 \mathcal{O}_F)$.

In [42], Qin described the method in [39] and [40] for calculating the 4-rank of the tame kernel of a quadratic number field by introducing sign matrices via Legendre symbols. Later in [55] and [56], Qin, Yin and Zhu exploited the same approach to determine the possible 4-ranks of the tame kernel for each type of quadratic fields. For each type they give a minimum and maximum possible value of $r_4$ and prove that the minimum, maximum and all intermediate values occur infinitely often.

In [41], Qin gave the necessary and sufficient conditions for an element of order two in $K_2 \mathcal{O}_F$ of a quadratic number field $F$ to be a fourth power in $K_2 \mathcal{O}_F$. This result also gave an effective method to compute the 8-rank of $K_2 \mathcal{O}_F$ for any quadratic number field. One can find the table of the 8-rank of $K_2 \mathcal{O}_F$ for any quadratic number field whose discriminant has only one odd prime divisor in [41].

Later in [43], Qin gave an alternative expression of the main result of [41] on the 8-rank of $K_2 \mathcal{O}_F$ and give more application. The 8-ranks of $K_2 \mathcal{O}_F$ for all quadratic number fields whose discriminants have exactly two odd prime divisors are completely determined. And also the Tate kernel of any imaginary quadratic number field $F$ with the 8-rank of $K_2 \mathcal{O}_F = 0$ has been given explicitly. If $F$ is a real quadratic number fields and both $-1$ and $-2$ are not in $NF = \text{Norm}_{F/\mathbb{Q}}(F^\times)$, then the 16-rank of $K_2 \mathcal{O}_F$ is determined in some cases where the 8-rank of $K_2 \mathcal{O}_F \neq 0$.

If $p$ is an odd prime, then the results on the $p$-Sylow subgroup of $K_2 \mathcal{O}_F$ are much less than that on the 2-Sylow subgroup of $K_2 \mathcal{O}_F$. One can see [6], [7], [9], [22], [27], [44] and [45] etc for results on the $p$-Sylow subgroup of $K_2 \mathcal{O}_F$. In [45], Qin established a reflection theorem for any odd prime $p$, which for $p = 3$ is Scholz Reflection Theorem, and gave some $p^n$-rank of $K_2 \mathcal{O}_F$ formulae.

We use the following notation: Let $F$ be a number field with $\mathcal{O}_F$ the ring of integers in $F$. For any integer $n$, $(K_2 \mathcal{O}_F)^n = \{ \alpha \in K_2 \mathcal{O}_F \mid \alpha = \beta^n \text{ for some } \beta \in K_2 \mathcal{O}_F \}$. For an integer $d \neq 0$, the set $S(d)$ is defined to be $\{ \pm 1, \pm 2 \}$ if $d > 0$ or $\{ 1, 2 \}$ if $d < 0$. For any abelian group $A$, let $2A = \{ \alpha \in A \mid \alpha^2 = 1 \}$. Let $p$ be a prime and $\mathbb{Q}_p$ the field of $p$-adic numbers. We shall use $\left( \frac{a, b}{p} \right)$ for the
Hilbert symbol with order 2 over \( \mathbb{Q}_p \), and \( v_p(\cdot) \) for the discrete valuation on \( \mathbb{Q}_p \).

The notation \((a, b) \equiv 1\) means that integers \(a\) and \(b\) have no common odd divisor.

2. The \( p \)-rank of \( K_2 \mathcal{O}_F \)

Let \( F \) be a number field, \( l \) a prime number, \( \mu_l \) the group of \( l \)-th roots of unity in the separable closure of \( F \), \( E = F(\mu_l) \), \( G = \text{Gal}(E/F) \). Then we have the following homomorphism:

\[
\tilde{\gamma} : \mu_l \otimes E^\times \rightarrow iK_2 E
\]

\[
z \otimes a \mapsto \{z, a\}, \text{ where } z \in \mu_l, a \in E^\times.
\]

If \( \tilde{\gamma} \) is restricted to \((\mu_l \otimes E^\times)^G\) the \( G \)-fixed subgroup of \( \mu_l \otimes E^\times \), then the image of \( \tilde{\gamma} \) is contained in the image of the natural map \( iK_2 E \rightarrow iK_2 \mathcal{E} \). Note that the degree of \( E/F \) is prime to \( l \). So \( iK_2 E \rightarrow iK_2 \mathcal{E} \) is injective. Hence we have a homomorphism

\[
\gamma : (\mu_l \otimes E^\times)^G \rightarrow iK_2 F.
\]

In 1976, Tate proved the following theorem.

**Theorem 2.1 (Tate, [51]).** With notations as above. Let \( r_2 \) be the number of complex places of \( F \). Let \( \varepsilon = 1 \) if \([F(\mu_l) : F] \leq 2\), and let \( \varepsilon = 0 \) otherwise. Then the map \( \gamma \) is surjective, and the kernel of \( \gamma \) is an elementary abelian group of order \( l^{r_2 + \varepsilon} \). In particular, if \( F \) contains a primitive \( l \)-th root of unity \( z \) and \( \Delta \) is the group of elements \( a \in F^\times \) such that \( \{z, a\} = 1 \), then every element of \( iK_2 F \) is of form \( \{z, b\} \) for some \( b \in F^\times \). Moreover, we have \((\Delta : (F^\times)^l) = l^{r_2 + 1} \).

The group \( \Delta \) in the above theorem is now called the Tate kernel of \( F \).

For a number field \( F \), let \( \text{Cl}(F) \) be its class group, and let \( \text{Cl}(F) \) be the subgroup of \( \text{Cl}(F) \) generated by classes containing prime ideals dividing \( l \). Let \( j_l = r_l(\text{Cl}(F)/\text{Cl}(F)) \) be the \( l \)-rank of \( \text{Cl}(F)/\text{Cl}(F) \). Let \( t_l \) be the number of finite places dividing \( l \).

By Tate’s Theorem, one can get the following Corollary.

**Corollary 2.2 (Keune, [27]).** Let \( l \) be a prime number. Suppose \( F \) contains a primitive \( l \)-th root of unity \( z \). Then we have the following formulae for \( p \)-rank

\[
r_p(K_2 \mathcal{O}_F) = j_l + t_l - 1, \text{ } (p \text{ odd})
\]

\[
r_2(K_2 \mathcal{O}_F) = j_2 + t_2 + r_1 - 1.
\]

Note that \( r_1 \) is the number of real places of \( F \).

Note that only in the above Corollary, \( r_2 \) denotes the 2-rank. While in other cases, \( r_2 \) always means the number of complex places of \( F \). The second formula of the above corollary was also proved earlier by Browkin in [5].

In case of quadratic number fields, Browkin and Schinzel proved in 1982 the following theorem. Recall that \( NF = \text{Norm}_{F/\mathbb{Q}}(F^\times) \).

**Theorem 2.3 (Browkin and Schinzel, [4]).** Let \( F = \mathbb{Q}(\sqrt{d}) \) be a quadratic number field, where \( d \) is a square free integer. Then \( 2K_2 \mathcal{O}_F \) can be generated by \((-1, m) \), \( m \mid d \), together with \((-1, u_i + \sqrt{d}) \) if \((-1, \pm 2) \cap NF \neq \emptyset \), where \( u_i \in \mathbb{Z} \) such that \( d = u_i^2 - c_i u_i^2 \) for some \( w_i \in \mathbb{Z} \) and \( c_i \in \{-1, \pm 2\} \cap NF \).

In 1985, by applying the odd part of Birch-Tate Conjecture, which follows from the Main Conjecture proved by Mazur-Wiles, Browkin obtained that
Theorem 2.4 (Browkin, [6]). If $d > 0$ is the discriminant of the field $F = \mathbb{Q}(\sqrt{d})$, then $3|\# K_2 \mathcal{O}_F$ if and only if the class number of the field $\mathbb{Q}(\sqrt{-3d})$ is divisible by 3.

For general results on the $p$-primary part of the tame kernel of number fields, one can see [51] and [27]. Based on his numerical computations, Gangl proposed in [16] some conjectures, which in the case $p = 3$, relate the divisibility of order of the tame kernel of imaginary quadratic number fields by 3 or 9 to the divisibility of class numbers of the same imaginary quadratic number fields by 3. Assuming Lichtenbaum’s conjecture, Browkin and Gangl gave a list of structures of the tame kernels and the wild kernels of imaginary quadratic number fields with discriminants larger than $-5000$ in [8].

In 1992, Browkin studied the $p$-rank of the tame kernels of quadratic number fields by the reflection theorem in [7]. He proved two of Gangl’s conjectures in [16].

Theorem 2.5 (Browkin, [7]). Let $d$ be a negative square free integer, $F = \mathbb{Q}(\sqrt{d})$. Then

1. $3|\# K_2 \mathcal{O}_F$ implies $3|\# \text{Cl}(F)$, for $d \not\equiv 3 \pmod{9}$.
2. $5|\# K_2 \mathcal{O}_F$ implies $5|\# \text{Cl}(\mathbb{Q}(\sqrt{5d}))$.

The remaining part of Gangl’s conjectures is proved by Guo and Qin ([22], using the logarithmic class group), Qin ([45], using the reflection theorems, class field theory and some new formulae for the $p$-rank of tame kernels). More precisely, we proved that

Theorem 2.6 ([22], [45]). If 9 divides the cardinality of $K_2 \mathcal{O}_{\mathbb{Q}(\sqrt{-9k-3})}$, then 3 divides the class number of $\mathbb{Q}(\sqrt{-9k-3})$, where $k$ is a positive integer and $3k+1$ is square free.

What was proved in [41] is quite general. More precisely, for any odd prime $p$, we consider the $p$-divisibility and also the $p^n$-divisibility of the order of $K_2 \mathcal{O}_F$. The result gives a general picture of the special phenomenon for prime 3 observed by Gangl.

In [9], Browkin developed some ideas for estimating the tame kernels of cyclic cubic fields. Wu extended in [54] the methods of Browkin to cyclic quintic fields. In [11], Cheng proved that for a fixed positive integer $m$ there exist infinitely many pure cubic fields whose 3-rank of the tame kernel equal to $m$. In [29], Li and Qin proved the following density result on the tame kernels of the pure cubic number fields.

Theorem 2.7 (Li and Qin, [29]). Let $X_{11} = \{m : m = p_1 p_2, p_1 \equiv 1 \mod{3}, p_2 \equiv 1 \mod{3}, p_1 p_2 \not\equiv \pm 1 \mod{9}, \left(\frac{p_2}{p_1}\right)_3 = 1\}$

and

$X_{12} = \{m : m = p_1 p_2, p_1 \equiv 1 \mod{3}, p_2 \equiv 2 \mod{3}, p_1 p_2 \not\equiv \pm 1 \mod{9}, \left(\frac{p_2}{p_1}\right)_3 = 1\}$. 

Then for the fields \( \mathbb{Q}(\sqrt[3]{p_1 p_2}) \), 3-rank 2 and 1 each appear with density \( \frac{1}{3} \) and \( \frac{2}{3} \) respectively in \( X_{12} \). And for the fields \( \mathbb{Q}(\sqrt[3]{p_1 p_2}) \), 3-rank 3, 2, and 1 each appear with density \( \frac{1}{27}, \frac{14}{27}, \) and \( \frac{4}{9} \) respectively in \( X_{11} \).

3. The 4-rank of \( K_2 \mathcal{O}_F \)

Since by Theorem 2.3, \( \{-1, m\} \) and \( \{-1, m(u + \sqrt{d})\} \) (if \( 2 \in \text{NF} \)) are the generators of \( K_2 \mathcal{O}_F \). The following theorem determines the 4-rank of \( K_2 \mathcal{O}_F \) for arbitrary quadratic number fields.

**Theorem 3.1 (Qin).** Let \( F = \mathbb{Q}(\sqrt{d}), d \in \mathbb{Z} \) square-free. Suppose that \( m \mid d \) \((m > 0 \text{ if } d > 0)\) and write \( d = u^2 - 2w^2 \) with \( u, w \in \mathbb{Z} \) (we take \( u > 0 \) if \( d > 0 \)) if \( 2 \in \text{NF} \). Then \( \{-1, m\} \in K_2 \mathcal{O}_F^2 \) if and only if one can find an \( \varepsilon \in S(d) \) such that for any odd prime \( p \mid d \),

\[
\left( \frac{-d, m}{p} \right) = \left( \frac{\varepsilon}{p} \right),
\]

and \( \{-1, m(u + \sqrt{d})\} \in K_2 \mathcal{O}_F^2 \) if and only if one can find a \( \delta \in S(d) \) such that for any odd prime \( p \mid d \),

\[
\left( \frac{-d, m}{p} \right) = \left( \frac{\delta(u + w)}{p} \right).
\]

The above theorem was proved in [39] and [40] and this version was presented in [43]. In [38], [39] and [40], Qin gave tables of the 4-ranks of tame kernels of \( K_2 \mathcal{O}_F \), where the number of odd prime factors of \( d \) is less than or equal to 3.

By Theorem 3.1, one can relate the 4-rank of \( K_2 \mathcal{O}_F \) with the 4-rank of the class group of \( \mathcal{O}_F \). See [3] and [57] for such relations. Since the density of 4-rank of class group of \( \mathcal{O}_F \) is already known, we can get the density of the 4-rank of \( K_2 \mathcal{O}_F \) (one can see [57], [58] and [20] for details).

Let \( D \) be a fundamental discriminant, i.e., the discriminant of some quadratic number field. Let

\[
g_r(D) = \begin{cases} 
1, & \text{if } r(4K_2 \mathcal{O}_{\mathbb{Q}(\sqrt{D})}) = r; \\
0, & \text{otherwise.}
\end{cases}
\]

Then the density of real quadratic number fields \( F \) with \( r(4K_2 \mathcal{O}_F) = r \), in the set of all real quadratic number fields is

\[
d_r^+ = \lim_{x \to \infty} \frac{\sum_{0 < D < x} g_r(D)}{\sum_{0 < D < x} 1}.
\]

And the density of imaginary quadratic number fields \( E \) with \( r(4K_2 \mathcal{O}_E) = r \) in the set of all imaginary quadratic number fields is

\[
d_r^- = \lim_{x \to \infty} \frac{\sum_{0 < -D < x} g_r(D)}{\sum_{0 < -D < x} 1}.
\]
Theorem 3.2 (Guo, [20]). Let \( r \) be an integer. Then
\[
d_{0}^{+} = 0; \\
d_{r}^{+} = \frac{2^{-r(r-1)} \prod_{k=1}^{\infty} (1 - 2^{-k})}{(1 - 2^{-r}) \prod_{k=1}^{r-1} (1 - 2^{-k})^2}, \quad \text{if } r \geq 1; \\
d_{r}^{-} = 2^{-r^2} \cdot \frac{\prod_{k=1}^{\infty} (1 - 2^{-k})}{\prod_{k=1}^{r} (1 - 2^{-k})^2}, \quad \text{if } r \geq 0.
\]

If the number of prime factors of the discriminant is small, the density of 4-ranks can be found in [35], [36], [37] and [10].

Osburn proved the following theorems.

Theorem 3.3 (Osburn, [35]). Let \( p \equiv 7 \mod 8 \) be a fixed prime and let
\[
\Omega = \{ l \text{ rational prime} | l \equiv 1 \mod 8 \text{ and } \left( \frac{l}{p} \right) = 1 \}.
\]

Then for the fields \( \mathbb{Q}(\sqrt{pl}) \) and \( \mathbb{Q}(\sqrt{2pl}) \), 4-rank 1 and 2 each appear with natural density 1/2 in \( \Omega \). For the fields \( \mathbb{Q}(\sqrt{-pl}) \) and \( \mathbb{Q}(\sqrt{-2pl}) \), 4-rank 0 and 1 each appear with natural density 1/2 in \( \Omega \).

Theorem 3.4 (Osburn, [36]). Let \( p \) be a fixed prime and let
\[
A_p = \{ l \text{ rational prime} | l \equiv 1 \mod 8 \text{ and } \left( \frac{l}{p} \right) = 1 \}
\]
\[
B_p = \{ l \text{ rational prime} | l \equiv 1 \mod 8 \text{ and } \left( \frac{l}{p} \right) = -1 \}.
\]

If \( p \equiv 1 \mod 8 \), then for the fields \( \mathbb{Q}(\sqrt{pl}) \), 4-rank 1 and 2 each appear with natural density 3/4 and 1/4 in \( A \). For the fields \( \mathbb{Q}(\sqrt{-pl}) \), 4-rank 1 and 2 each appear with natural density 1/2 in \( A \). For the fields \( \mathbb{Q}(\sqrt{pl}) \), 4-rank 0 and 1 each appear with natural density 1/2 in \( B \).

Later Cheng proved the following.

Theorem 3.5 (Cheng, [10]). Let \( A_p \) and \( B_p \) be the same as in the above theorem. Then

1. Assume \( p \equiv 1 \mod 8 \). Then for real quadratic number field \( \mathbb{Q}(\sqrt{2pl}) \), 4-rank 1 and 2 each appear with natural density \( \frac{3}{4} \) and \( \frac{1}{4} \) in \( A_p \). For the imaginary number field \( \mathbb{Q}(\sqrt{-2pl}) \), 4-rank 1 and 2 each appear with natural density \( \frac{3}{4} \) and \( \frac{1}{4} \) in \( A_p \).

2. Assume \( p \equiv 7 \mod 8 \). Then for the imaginary number field \( \mathbb{Q}(\sqrt{-2pl}) \), 4-rank 0 and 1 each appear with natural density \( \frac{1}{2} \) in \( A_p \). For the imaginary number field \( \mathbb{Q}(\sqrt{-2pl}) \), 4-rank 0 and 1 each appear with natural density \( \frac{1}{2} \) in \( B_p \).

Theorem 3.6 (Osburn, [37]). Let
\[
X = \{ d | d = p_1 p_2 p_3, \ p_i \equiv 1 \mod 8 \},
\]
where \( p_i \) are distinct rational primes. Then for the fields \( \mathbb{Q}(\sqrt{p_1p_2p_3}) \), 4-rank 0, 1, 2 and 3 each appear with natural density \( \frac{1}{4}, \frac{17}{64}, \frac{13}{64} \) and \( \frac{1}{64} \) in \( X \).

Theorem 3.1 gives an efficient method to compute the 4-rank of \( K_2\mathcal{O}_F \). In [42], Qin explained this method via the sign matrices. Let \( d \in \mathbb{N} \) square-free and let \( d = 2^\sigma p_1 \ldots p_n \) be the prime factorization, where \( \sigma = 0 \) or 1. For \( j = 1, 3, 5, 7 \) we let \( m_j \) denote the number of \( p_i \)'s which are \( \equiv j \mod 8 \), and we call \( 2^\sigma (m_1, m_3, m_5, m_7) \) the type of \( d \). The results of [42] are expressed in terms of the type of the square free integer \( d \). In the real quadratic case, Qin characterizes those types for which the 4-rank of the tame kernel is always positive and shows that for each other type there exists both a set of \( d \) of positive density for which the 4-rank is 0 and a set of \( d \) of positive density for which the 4-rank is positive. Similar results were proved for the imaginary quadratic number fields.

Later in [55], Qin, Yin and Zhu exploited the same method to determine the possible 4-ranks of the tame kernel for each type of real quadratic field. For each type they gave the minimum and maximum possible value of \( r_4 \) and prove (modulo a plausible technical assumption in a few cases) that all intermediate values occur infinitely often. In [56], Qin, Yin and Zhu applied this method to prove that the similar assertion holds for imaginary quadratic fields \( F \).

Let \( d \) be a square free integer, \( D \) the discriminant of \( \mathbb{Q}(\sqrt{d}) \). It is natural to raise the following conjecture.

**Conjecture 3.7** Let \( \sigma = 0 \) or 1, \( m_1, m_3, m_5, m_7 \) fixed non negative integers. Let \( T_+ = \{ D | 0 < D < x, \text{ and the type of } d = 2^\sigma (m_1, m_3, m_5, m_7) \} \). Let \( r_{\text{min}} \) and \( r_{\text{max}} \) be the minimum and maximum possible value of \( r_4 \) corresponding to the type, which were given in [42]. Then the density of real quadratic number fields \( F \) with \( r_4(K_2\mathcal{O}_F) = r \), \( r_{\text{min}} \leq r \leq r_{\text{max}} \), in the set of all real quadratic number fields of type \( 2^\sigma (m_1, m_3, m_5, m_7) \)

\[
d^+_{r_4}(2^\sigma (m_1, m_3, m_5, m_7)) := \lim_{x \to \infty} \frac{\sum_{D \in T_+} g_r(D)}{\sum_{D \in T_+} 1}
\]

is positive. The similar assertion holds also for imaginary quadratic fields.

4. The 8-rank of \( K_2\mathcal{O}_F \)

Let \( F = \mathbb{Q}(\sqrt{d}) \) be a quadratic field, where \( d \in \mathbb{Z} \) is square-free. Recall from [2] that \( 2K_2\mathcal{O}_F \) can be written as the forms \{\(-1, m\), \( m \mid d \); together with \{-1, \( m(u_i + \sqrt{d}) \), if \{-1, \pm 2\} \cap NF \neq \emptyset \), where \( u_i \in \mathbb{Z} \) such that \( u_i^2 - d = c_i w_i^2 \) for some \( w_i \in \mathbb{Z} \) and \( c_i \in \{-1, \pm 2\} \cap NF \).

If \( u_i^2 - d = -w_i^2 \) or \(-2w_i^2 \), then \{-1, \( m(u_i + \sqrt{d}) \) \( \notin (K_2\mathcal{O}_F)^4 \.

On the other hand, we know that a necessary condition for \{-1, \( m \) \( \in (K_2\mathcal{O}_F)^4 \) is that there is an \( \epsilon \in \{1, 2\} \) such that

\[
(4.1) \quad \epsilon mZ^2 = X^2 + dY^2
\]
is solvable. And a necessary condition for \{-1, \( m(u + \sqrt{d}) \) \( \in (K_2\mathcal{O}_F)^4 \) is that

\[
(4.2) \quad m(u + w)Z^2 = X^2 + dY^2
\]
is solvable.
For a square-free integer $d$ and $i = 1, 3, 5, 7$, denote by $d_i$ the product of all prime divisors of $d$ which are $\equiv i \pmod{8}$ ($d_i = 1$ if $d$ has no prime divisor $\equiv i \pmod{8}$).

We let $\sigma(l) = 1$ or $0$ according to $l \mid m_3$ or not. In view of Theorem 3.1, the following theorem determines the 8-rank of $K_3 \mathcal{O}_F$ for arbitrary quadratic number fields completely.

**Theorem 4.1 (Qin, [41], [43]).** Let $d$ be a square-free integer and $F = \mathbb{Q}(\sqrt{d})$, and let $m \mid d$. Write $m = \pm m_1 m_3 m_5 m_7$ with $m_i \mid d_i$ for $i = 1, 3, 5, 7$. Assume that (4.1) is solvable and let $X_m, Y_m, Z_m \in \mathbb{N}$ with $(X_m, Y_m) = 1$ and $(Z_m, d)^2 \equiv 1$ is a solution of (4.1).

(A) Suppose that $2 \notin NF$. Then $\{-1, m\} \in (K_3 \mathcal{O}_F)^4$ if and only if for $i = 1, 3, 5, 7$, there are $h_i \mid d_i$, in particular, $h_i = 1$ is permitted, and $\varepsilon \in \{ \pm 1, \pm 2 \}$ such that for any odd prime $l \mid d$,

$$\left( \frac{d, m_3 h_1 h_5}{l} \right) \left( \frac{-2^{\sigma(l)} d, m_5 h_3 h_7}{l} \right) = \left( \frac{\varepsilon Z_m}{l} \right).$$

(B) Suppose that $2 \in NF$.

(i) Then $\{-1, m\} \in (K_3 \mathcal{O}_F)^4$ if and only if for $i = 1, 3, 5, 7$, there are $h_i \mid d_i$ ($h_i = 1$ is permitted) and $\delta \in \{ \pm 1 \}$ such that for any odd prime $l \mid d$,

$$\left( \frac{d, h_1}{l} \right) \left( \frac{-d, h_7}{l} \right) = \left( \frac{\delta Z_m}{l} \right),$$

or

$$\left( \frac{d, h_1}{l} \right) \left( \frac{-d, h_7}{l} \right) = \left( \frac{(\delta u + w)Z_m}{l} \right).$$

(ii) Assume that (4.2) is solvable and let $X_{mv}, Y_{mv}, Z_{mv} \in \mathbb{N}$ with $h = Y_{mv}, g = \frac{X_{mv} - wY_{mv}}{u + w} \in \mathbb{Z}$, $(g, h) = 1$ and $(Z_{mv}, dw) = 1$ be a solution. Then $\{-1, m(u + \sqrt{d})\} \in (K_3 \mathcal{O}_F)^4$ if and only if for $i = 1, 3, 5, 7$, there are $h_i \mid d_i$ ($h_i = 1$ is permitted) and $\eta \in \{ \pm 1 \}$ such that for any odd prime $l \mid d$,

$$\left( \frac{d, h_1}{l} \right) \left( \frac{-d, h_7}{l} \right) = \left( \frac{\eta u Z_{mv}}{l} \right),$$

or

$$\left( \frac{d, h_1}{l} \right) \left( \frac{-d, h_7}{l} \right) = \left( \frac{\eta (u + w) u Z_{mv}}{l} \right).$$

By Theorem 4.1, in order to compute the 8-rank of $K_3 \mathcal{O}_F$, one needs to solve some Diophantine equations. There are $2^{r_4+1}$ (resp., at most $2^{r_4}$) such equations in the imaginary (resp., real) case. The following theorem permits us to consider only $r_4 + 1$ (resp., at most $r_1$) equations in the imaginary (resp., real) case.

**Theorem 4.2 (Qin, [43]).** Let $d$ be a square-free integer and $m \mid d, n \mid d$. And let $(m, n) = c$. Suppose that there are $\epsilon_m, \epsilon_n \in \{ 1, 2 \}$ such that $\epsilon_m mZ^2 = X^2 + dY^2$ and $\epsilon_n nZ^2 = X^2 + dY^2$ are solvable. Then $\epsilon_m \epsilon_n (m^n + c) Z^2 = X^2 + dY^2$ is solvable and we can choose solutions of these equations such that $Z_m, Z_n, Z_{mn/c^2}$ satisfy the assumption of Theorem 4.1 and for any odd prime $p \mid d$,

$$\left( \frac{Z_m}{p} \right) \left( \frac{Z_n}{p} \right) = \left( \frac{\varepsilon Z_{mn/c^2}}{p} \right).$$
for some $\varepsilon = 1$ or 2.

Suppose that $2 \in N\mathbb{Q}(\sqrt{d})$ and $r \mid d$. Suppose that $n(u + w)Z^2 = X^2 + dY^2$ and $r(u + w)Z^2 = X^2 + dY^2$ are solvable. Then $nr/(n + r)^2(u + w)Z^2 = X^2 + dY^2$ is solvable and we can choose solutions of these equations such that $Z_m, Z_{nv}, Z_{mn/c^2v}$ satisfy the assumption of Theorem 4.1 and for any odd prime

$$
\left( \frac{Z_m}{p} \right) \left( \frac{Z_{nv}}{p} \right) = \left( \frac{T_{mn/c^2v}}{p} \right)
$$

for some $\delta = 1$ or 2 and

$$
\left( \frac{Z_{nv}}{p} \right) \left( \frac{Z_{rw}}{p} \right) = \left( \frac{\eta_{nr/(n + r)^2}}{p} \right)
$$

for some $\eta = 1$ or 2.

If the number of odd prime factors of the discriminant of quadratic number field $F$ is less than 2, then $2^n$-rank ($n \leq 3$) of $K_2\mathcal{O}_F$ is explicitly given in [38], [39], [40], [41], [42], [43]. In some cases, even the $2^k$-rank of $K_2\mathcal{O}_F$ is explicitly given.

In general case, Qin proposed the following Conjecture in [43].

**Conjecture 4.2:** Let $k \geq 2$ and $n \in \mathbb{N}$. Given $k - 1$ integers $r_4, r_8, ..., r_{2k}$ satisfying

$$
n \geq r_4 \geq r_8 \geq \cdots \geq r_{2k} \geq 0.
$$

Then there exist infinitely many quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ such that $d > 0$ is square-free has exactly $n$ prime divisors, any of which $\equiv 1 \pmod{8}$ and the $2^j$-rank of $K_2\mathcal{O}_F = r_2k \geq 2$ ($2 \leq j \leq k$).

The same assertion should be true for $F = \mathbb{Q}(\sqrt{d})$ with $d = -d'$ or $d = 2d'$ or $d = -2d'$, where $d'$ has exactly $n$ prime divisors, any of which $\equiv 1 \pmod{8}$.

In [42], Qin proved that above conjecture is true for $k = 2$ and $n - 1 \geq r_4 \geq 0$. With some assumption, it is also proved that the Conjecture is true for $k = 2$.

In [21], we proved the following theorem.

**Theorem 4.3.** For any finite abelian group $G$ of exponent 8, there are infinitely many imaginary quadratic fields $E$ such that

$$K_2\mathcal{O}_E/(K_2\mathcal{O}_E)^G \cong G;
$$

and for any finite abelian group $H$ of exponent 8 with $rk_2H \geq 2 + rk_4H$, there are infinitely many real quadratic fields $F$ such that

$$K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^G \cong H.
$$

This result can be seen as the $K_2$-analogue of Morton and Stevenhagen’s result in [32], [33], [34] and [49].

**References**


[34] P. Morton, Density results for the 2-classgroups and fundamental units of real quadratic fields, Studia Scientiarum Mathematicarum Hungarica 17 (1982), 21–43.
