Combinatorics

南京大学
尹一通
Course Info

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• Office hour: 804, Tuesday, 2pm-4pm

• course homepage:
  • http://tcs.nju.edu.cn/wiki/
Textbook

van Lint and Wilson,
A course in Combinatorics,
2nd Edition.

Jukna,
Extremal Combinatorics: with applications in computer science,
2nd Edition.
Reference Books

Stanley, *Enumerative Combinatorics, Volume 1*

Graham, Knuth, and Patashnik, *Concrete Mathematics: A Foundation for Computer Science*
Reference Books

Aigner and Ziegler.  
*Proofs from THE BOOK.*

Alon and Spencer.  
*The Probabilistic Method.*

Cook, Cunningham, Pulleyblank, and Schrijver.  
*Combinatorial Optimization.*
Combinatorics

- **Enumeration (counting):** How many solutions to these constraints?
- **Existence:** Does a solution exist?
- **Extremal:** How large/small a solution can be to preserve/avoid certain structure?
- **Ramsey:** When a solution is sufficiently large, some structure must emerge.
- **Optimization:** Find the optimal solution.
- **Construction (design):** Construct a solution.

**Solution:** combinatorial object
**Constraint:** combinatorial structure

Combinatorial $\approx$ discrete finite
Tools (and prerequisites)

- Combinatorial (elementary) techniques;
- Algebra (linear & abstract);
- Probability theory;
- Analysis (calculus).
 Enumeration
(counting)

How many ways are there:

• to rank $n$ people?
• to assign $m$ zodiac signs to $n$ people?
• to choose $m$ people out of $n$ people?
• to partition $n$ people into $m$ groups?
• to distribute $m$ yuan to $n$ people?
• to partition $m$ yuan to $n$ parts?
• ... ...
The Twelvefold Way

Stanley, *Enumerative Combinatorics, Volume I*

Gian-Carlo Rota (1932-1999)
The twelvefold way

\[ f : N \rightarrow M \quad |N| = n, \quad |M| = m \]

<table>
<thead>
<tr>
<th>elements of ( N )</th>
<th>elements of ( M )</th>
<th>any ( f )</th>
<th>1-1</th>
<th>on-to</th>
</tr>
</thead>
<tbody>
<tr>
<td>distinct</td>
<td>distinct</td>
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<td>identical</td>
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Knuth’s version (in *TAOCP* vol. 4A)

$n$ balls are put into $m$ bins

<table>
<thead>
<tr>
<th>balls per bin:</th>
<th>unrestricted</th>
<th>$\leq 1$</th>
<th>$\geq 1$</th>
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<tbody>
<tr>
<td>$n$ distinct balls, $m$ distinct bins</td>
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Counting (labeled) trees

“How many different trees can be formed from \( n \) distinct vertices?”
Chapter 30

Arthur Cayley

One of the most beautiful formulas in enumerative combinatorics concerns the number of labeled trees. Consider the set $N = \{1, 2, \ldots, n\}$. How many different trees can we form on this vertex set? Let us denote this number by $T_n$. Enumeration "by hand" yields $T_1 = 1$, $T_2 = 1$, $T_3 = 3$, $T_4 = 16$, with trees shown in the table:

<p>| | | |</p>
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</table>

Note that we consider labeled trees, that is, although there is only one tree of order 3 in the sense of graph isomorphism, there are 3 different labeled trees obtained by marking the inner vertex 1, 2, or 3. For $n = 5$ there are three non-isomorphic trees:

For the first tree there are clearly 5 different labelings, and for the second and third there are $5! / 2 = 60$ labelings, so we obtain $T_5 = 125$. This should be enough to conjecture $T_n = n^{n-2}$, and that is precisely Cayley's result.

Theorem. There are $n^{n-2}$ different labeled trees on $n$ vertices.

Cayley's formula:

There are $n^{n-2}$ trees on $n$ distinct vertices.

Arthur Cayley (1821-1895)
Algorithmic Enumeration

 enumeration algorithm:
 for $i = 1, 2, 3, \ldots n^{n-2}$
 output the $i$-th tree;

 counting algorithm:

 input: undirected graph $G(V, E)$

 $t(G)$ : “The number of different spanning trees of $G(V, E)$.”
Graph Laplacian

Graph $G(V,E)$

adjacency matrix $A$

$$A(i, j) = \begin{cases} 1 & \{i, j\} \in E \\ 0 & \{i, j\} \notin E \end{cases}$$

diagonal matrix $D$

$$D(i, j) = \begin{cases} \deg(i) & i = j \\ 0 & i \neq j \end{cases}$$

diagonal matrix $D$

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

graph Laplacian $L$

$$L = D - A$$
Let $L_{i,i}$ denote the submatrix of $L$ by removing $i$th row and $i$th column.

$t(G)$ : number of spanning trees in $G$

**Kirchhoff’s Matrix-Tree Theorem:**

$$\forall i, \quad t(G) = \det(L_{i,i})$$
Bipartite Perfect Matching

bipartite graph

\[
G([n],[n],E)
\]

permutation \( \pi \) of \([n]\)

\( n \times n \) matrix \( A \):

\[
A_{i,j} = \begin{cases} 
1 & (i, j) \in E \\
0 & (i, j) \not\in E 
\end{cases}
\]

perfect matchings

s.t. \((i, \pi(i)) \in E\)

\# of P.M. in \( G \)

\[
= \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)}
\]
Permanent

$n \times n$ matrix $A$:

$$\text{perm}(A) = \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i, \pi(i)}$$

**#P-hard** to compute

determinant:

$$\text{det}(A) = \sum_{\pi \in S_n} (-1)^{r(\pi)} \prod_{i \in [n]} A_{i, \pi(i)}$$

poly-time by Gaussian elimination
Ryser’s formula

\[ \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i, \pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i, j} \]

\(O(n!)\) time \hspace{1cm} \(O(n2^n)\) time

PIE (Principle of Inclusion-Exclusion):

\[ \sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases} \]
PIE
(Principle of Inclusion-Exclusion)

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
Inversion

$V$: $2^n$-dimensional vector space of all mappings

$f : 2^n \rightarrow \mathbb{N}$

linear transformation $\phi : V \rightarrow V$

$\forall S \subseteq [n], \quad \phi f(S) \triangleq \sum_{\substack{T \supseteq S \\
T \subseteq [n]}} f(T)$

then its inverse:

$\forall S \subseteq [n], \quad \phi^{-1} f(S) = \sum_{\substack{T \supseteq S \\
T \subseteq [n]}} (-1)^{|T \setminus S|} f(T)$
Fibonacci number

\[ F_n = \begin{cases} 
    F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\
    1 & \text{if } n = 1 \\
    0 & \text{if } n = 0.
\end{cases} \]

\[ F_n = \frac{1}{\sqrt{5}} \left( \phi^n - \hat{\phi}^n \right) \]

\[ \phi = \frac{1 + \sqrt{5}}{2} \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2} \]

by generating functions ...
**Quicksort**

**input:** an array $A$ of $n$ numbers

<table>
<thead>
<tr>
<th>Qsort($A$):</th>
</tr>
</thead>
<tbody>
<tr>
<td>choose a <strong>pivot</strong> $x = A[1]$;</td>
</tr>
<tr>
<td>partition $A$ into $L$ with all $L[i] &lt; x$, $R$ with all $R[i] &gt; x$;</td>
</tr>
<tr>
<td>Qsort($L$) and Qsort($R$);</td>
</tr>
</tbody>
</table>

**Complexity:** number of comparisons

**worst-case:** $\Theta(n^2)$

**average-case:** ?
Qsort(A):

- choose a pivot $x = A[1]$;
- partition $A$ into $L$ with all $L[i] < x$, $R$ with all $R[i] > x$;
- Qsort($L$) and Qsort($R$);

**pivot:** the $k$-th smallest number in $A$

- $|L| = k - 1$
- $|R| = n - k$

**Recursion:**

\[
T_n = \frac{1}{n} \sum_{k=1}^{n} (n - 1 + T_{k-1} + T_{n-k}) = 2n \ln n + O(n)
\]

$T_0 = T_1 = 0$
Counting with Symmetry

Rotation :

Rotation & Reflection:
Symmetries
Pólya’s Theory of Counting

George Pólya (1887-1985)
pattern inventory: (multi-variate) generating function

\[ F_G(y_1, y_2, \ldots, y_m) = \sum_{\bar{v} = (n_1, \ldots, n_m)} a_{\bar{v}} y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} \]

\[ n_1 + \cdots + n_m = n \]

\( a_{\bar{v}} \): # of config. (up to symmetry) with \( n_i \) many color \( i \)

Pólya's enumeration formula (1937):

\[ F_G(y_1, y_2, \ldots, y_m) = P_G \left( \sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \ldots, \sum_{i=1}^m y_i^n \right) \]

\[ \pi = \underbrace{\cdots}_{k \text{ cycles}} \underbrace{\cdots}_{\ell_k} \underbrace{\cdots}_{\ell_2} \underbrace{\cdots}_{\ell_1} \]

\[ M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^k x_{\ell_i} \]

cycle index: \[ P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n) \]
$F_{D_{20}} (r, q, l)$
Existing

Does there exist:

- a configuration satisfying this condition?
- a counterexample for this method?
- an efficient algorithm for this problem?
- a problem which is hard to solve in this computation model?
- ...

Circuit Complexity

Boolean function \[ f : \{0, 1\}^n \rightarrow \{0, 1\} \]

Boolean circuit
Theorem (Shannon 1949)

There is a boolean function $f : \{0, 1\}^n \to \{0, 1\}$ which cannot be computed by any circuit with $\frac{2^n}{3n}$ gates.

no constructive proof is known
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