The Socle of the Last Term in the Minimal Injective Resolution of a Gorenstein Module

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Abstract

Let $R$ be a left Noetherian ring, $S$ a right Noetherian ring and $RU$ a Gorenstein module with $S = \text{End}(RU)$. If the injective dimensions of $RU$ and $US$ are finite, then the last term in the minimal injective resolution of $RU$ has an essential socle.

1 Introduction

Recall that a left and right Noetherian ring is called Gorenstein if its left and right self-injective dimensions are finite. The following question still remains open.

\textbf{Question 1.1.} For a Gorenstein ring $R$, is the socle of the last term in the minimal injective resolution of $RU$ non-zero?

The answer to this question is positive in any case of the following

(1) $R$ is a left and right Artinian ring.

(2) The left and right self-injective dimensions of $R$ are at most 2 ([7, Theorem 4.5]).

(3) $R$ is an Auslander-Gorenstein ring ([5, Proposition 1.1]).

Furthermore, in the case (3) above, Iwanaga and Sato showed in [12, Theorem 6] that this socle is essential in the last term. As a natural generalization of Auslander’s $n$-Gorenstein rings, Huang introduced in [8] the notion of $n$-Gorenstein modules such that a left and right Noetherian ring $R$ is
Auslander’s \( n \)-Gorenstein if and only if it is \( n \)-Gorenstein as an \( R \)-module. Then Huang and Wang proved in [11, Theorem 3.1] that for left and right Noetherian rings \( R \) and \( S \) and a generalized tilting module \( _RU \) with \( S = \text{End}(RU) \), if \( _RU \) is \((n - 2)\)-Gorenstein with the injective dimensions of \( _RU \) and \( U_S \) being \( n \) (where \( n \) is a non-negative integer), then the socle of the last term in the minimal injective resolution of \( _RU \) is non-zero. In this paper we extend these results and prove the following

**Theorem 1.2.** Let \( R \) be a left Noetherian ring, \( S \) a right Noetherian ring and \( _RU \) a Gorenstein module with \( S = \text{End}(RU) \). If the injective dimensions of \( _RU \) and \( U_S \) are finite, then the last term in the minimal injective resolution of \( _RU \) has an essential socle.

In Section 2, we give some terminology and some preliminary results. In Section 3, we introduce the notion of Gorenstein modules. Let \( R \) be a left Noetherian ring, \( S \) a right Noetherian ring and \( _RU \) a Gorenstein module with \( S = \text{End}(RU) \) such that the injective dimensions of \( _RU \) and \( U_S \) are equal to \( n \). We first prove that \( \text{Ext}^i_{S^{op}}(N, U) \) is an Artinian left \( R \)-module for any finitely generated right \( S \)-module \( N \). Then we get that any non-zero submodule of the last term in the minimal injective resolution of \( _RU \) has a non-zero Artinian submodule. Theorem 1.2 follows from this result.

## 2 Preliminaries

Let \( R \) be an arbitrary associative ring with identity, and let \( \text{Mod } R \) be the category of left \( R \)-modules and \( \text{mod } R \) the category of finitely generated left \( R \)-modules. For a module \( M \) in \( \text{Mod } R \), we use \( \text{add}_R M \) to the full subcategory of \( \text{Mod } R \) consisting of modules isomorphic to direct summands of finite direct sums of copies of \( R \)M, and use \( \text{pd}_R M \), \( \text{id}_R M \) and \( \text{fd}_R M \) to denote the projective, injective and flat dimensions of \( M \) respectively. We use \( \text{gen}^*(R) \) to denote the full subcategory of \( \text{mod } R \) consisting of modules admitting a degreewise finite \( R \)-projective resolution.

**Definition 2.1.** ([17, 18]) A module \( _RU \) is called generalized tilting (sometimes it is also called Wakamatsu tilting, see [2, 13]), if the following conditions are satisfied.

1. \( _RU \in \text{gen}^*(R) \).
2. \( \text{Ext}_R^{>1}(U, U) = 0 \), that is, \( _RU \) is self-orthogonal.
3. There exists an exact sequence

   \[
   0 \rightarrow _RU \rightarrow U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow \cdots
   \]

   in \( \text{mod } R \) with all \( U_i \) in \( \text{add}_R U \), such that after applying the functor \( \text{Hom}_R(-, _RU) \) the sequence is still exact.
Let \( R \) and \( S \) be arbitrary associative rings with identity. Recall that a bimodule \( R_U S \) is called \textit{faithfully balanced} if \( R = \text{End}(U_S) \) and \( S = \text{End}(RU) \). By [18, Corollary 3.2], we have that \( R_U S \) is faithfully balanced and self-orthogonal with \( RU \in \text{gen}^*(R) \) and \( US \in \text{gen}^*(S) \) if and only if \( RU \) is generalized tilting with \( S = \text{End}(RU) \), and if and only if \( US \) is generalized tilting with \( R = \text{End}(US) \). Note that a faithfully balanced and self-orthogonal bimodule \( R_U S \) with \( RU \in \text{gen}^*(R) \) and \( US \in \text{gen}^*(S) \) is also called a \textit{semidualizing bimodule} (cf. [11]).

Let \( RU \) be a generalized tilting module with \( S = \text{End}(RU) \). We write \( (\cdot)^* := \text{Hom}(RU, \cdot) \) and \( \cdot^*: \text{Hom}(\cdot, RU) \).

We use  
\[
0 \to RU \to E_0 \to E_1 \to \cdots \to E_i \to \cdots
\]

to denote the minimal injective resolution of \( RU \) and \( K_i = \text{Ker}(E_i \to E_{i+1}) \) for any \( i \geq 0 \) (note: \( K_0 = RU \)), and use  
\[
0 \to US \to E'_0 \to E'_1 \to \cdots \to E'_i \to \cdots
\]

to denote the minimal injective resolution of \( US \). Following [8], we use \( \text{add-lim}\ RU \) (resp. \( \text{add-lim}\ US \)) to denote the full subcategory of \( \text{Mod}R \) (resp. \( \text{Mod}S^{op} \)) consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of \( RU \) (resp. \( US \)).

**Definition 2.2.** ([8]) For a module \( M \) in \( \text{Mod}R \), if there exists an exact sequence  
\[
\cdots \to U_n \to \cdots \to U_1 \to U_0 \to M \to 0
\]
in \( \text{Mod}R \) with all \( U_i \) in \( \text{add-lim}\ RU \), then we define \( \text{U-lim}\dim_R M = \inf\{n \mid \text{there exists an exact sequence}
\]
\[
0 \to U_n \to \cdots \to U_1 \to U_0 \to M \to 0
\]
in \( \text{Mod}R \) with all \( U_i \) in \( \text{add-lim}\ RU \}. \) We set \( \text{U-lim}\dim_R M \) infinity if no such an integer exists. For \( S^{op}\)-modules, we may define such a dimension similarly.

Let \( M \) be in \( \text{mod}R \) and \( i \geq 0 \). We say that the \textit{grade} of \( M \) with respect to \( U \), written \( \text{grade}_U M \), is at least \( i \) if \( \text{Ext}_R^{0 \leq j < i}(M, U) = 0 \). We say that the \textit{strong grade} of \( M \) with respect to \( U \), written \( \text{s.grade}_U M \), is at least \( i \) if \( \text{grade}_U X \geq i \) for any finitely generated \( R\)-submodule \( X \) of \( M \) (cf. [8]). The following result was proved in [8, Theorem 17.1.11] when \( R \) and \( S \) are two-sided Noetherian rings. Because the argument there remains valid in the setting here, we omit it.

**Theorem 2.3.** Let \( R \) be a left Noetherian ring, \( S \) a right Noetherian ring and \( RU \) a generalized tilting module with \( S = \text{End}(RU) \). Then for any \( n \geq 0 \), the following statements are equivalent.
(1) \( U\lim \dim_R E_i \leq i \) for any \( 0 \leq i \leq n - 1 \).

(2) \( \text{fd}_S E_i \leq i \) for any \( 0 \leq i \leq n - 1 \).

(3) \( \text{s.grade}_U \text{Ext}_R^i(M, U) \geq i \) for any \( M \in \text{mod} \, R \).

(4) \( \text{fd}_{S^{op}} E_i' \leq i \) for any \( 0 \leq i \leq n - 1 \).

(5) \( U\lim \dim_{S^{op}} E_i' \leq i \) for any \( 0 \leq i \leq n - 1 \).

(6) \( \text{s.grade}_{S^{op}} \text{Ext}_S^i(N, U) \geq i \) for any \( N \in \text{mod} \, S^{op} \).

If one of the equivalent conditions in Theorem 2.3 is satisfied, then \( R \, U \) (equivalent \( U \, S \)) is called \( n \)-Gorenstein ([8, 10]). So a left and right Noetherian ring \( R \) is (Auslander) \( n \)-Gorenstein ([12]) if and only if \( R \, R \) is \( n \)-Gorenstein, and if and only if \( R \, R \) is \( n \)-Gorenstein.

3 Main Results

In this section, we give the proof of Theorem 1.2. We begin with the following

Lemma 3.1. Let \( R \) be a ring and

\[
0 \to K \xrightarrow{f} M \to N \to 0
\]

an exact sequence in \( \text{Mod} \, R \) with \( N \neq 0 \) and \( f \) an essential monomorphism. Then \( \text{Ext}_R^1(X, K) \neq 0 \) for any non-zero \( R \)-submodule \( X \) of \( N \).

Proof. Let \( X \) be a non-zero \( R \)-submodule of \( N \) and \( \alpha : X \hookrightarrow N \) the inclusion. Then we have the following pull-back diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
| & & | \\
\downarrow & & \downarrow \\
0 & \to & K \\
| & & | \\
\downarrow & & \downarrow \\
W & \to & X & \to & 0 \\
\| & \| & \| & \| \\
\| & \| & \| & \| \\
\| & \| & \| & \| \\
\| & \| & \| & \| \\
M & \xrightarrow{f} & N & \to & 0 \\
\| & \| & \| & \| \\
\| & \| & \| & \| \\
\| & \| & \| & \| \\
\| & \| & \| & \| \\
Coker \alpha & \xrightarrow{=} & Coker \alpha & \to & 0
\end{array}
\]

We claim that the upper row does not split. Otherwise, if it splits, then \( K \) is isomorphic to a non-trivial direct summand of \( W \). So \( K \) is not isomorphic to an essential submodule of \( W \), and hence \( K \)
is not an essential submodule of $M$. It contradicts that $f$ is an essential monomorphism. The claim is proved. Thus we have $\text{Ext}^1_R(X,K) \neq 0$.

From now on, $R$ is a left Noetherian ring, $S$ is a right Noetherian ring and $RU$ is a generalized tilting module with $S = \text{End}(RU)$. By [9, Theorem 2.7], we have that $\text{id}_RU = \text{id}_{S^op}U$ provided both of them are finite.

**Lemma 3.2.** If $\text{id}_RU = n$, then for any non-zero $R$-submodule $X$ of $E_n$, we have $\text{Ext}^n_R(X,U) \neq 0$.

**Proof.** Because $\text{id}_RU = n$, we have the following exact sequence

$$0 \to K_{n-1} \to E_{n-1} \to E_n \to 0$$

in $\text{Mod } R$ with $K_{n-1} \to E_{n-1}$ an essential monomorphism. Let $X$ be a non-zero $R$-submodule of $E_n$. Then $\text{Ext}^1_R(X,K_{n-1}) \neq 0$ by Lemma 3.1. Thus we have $\text{Ext}^n_R(X,U) \cong \text{Ext}^1_R(X,K_{n-1}) \neq 0$.

By Lemma 3.2, we have the following

**Proposition 3.3.**

1. If $\text{id}_RU = n(\geq 1)$, then $E_0$ and $E_n$ have no isomorphic non-zero direct summands.

2. If $\text{id}_RU = \text{id}_{S^op}U = n$, then $\text{pd}_{S^*}E = n$ for any non-zero direct summand $E$ of $E_n$.

**Proof.** (1) Assume that $E \neq 0$ is an indecomposable direct summand of $E_0$ and $E$ is isomorphic to a direct summand of $E_n$. Then $E$ is the injective envelope of some finitely generated non-zero $R$-submodule $Y$ of $E$ by [14, Theorem 2.4]. Since $U$ is an essential submodule of $E_0$, we have $X := U \cap Y \neq 0$. From the exact sequence

$$0 \to X \to U \to U/X \to 0,$$

we get the following exact sequence

$$0 = \text{Ext}^n_R(U,U) \to \text{Ext}^n_R(X,U) \to \text{Ext}^{n+1}_R(U/X,U).$$

Because $\text{id}_RU = n$, we have $\text{Ext}^{n+1}_R(U/X,U) = 0$. It induces that $\text{Ext}^n_R(X,U) = 0$. On the other hand, because $0 \neq X < E \to E_n$, we have $\text{Ext}^n_R(X,U) \neq 0$ by Lemma 3.2. It is a contradiction.

(2) Let $E$ be a non-zero direct summand of $E_n$. Then $\text{Ext}^n_R(E,U) \neq 0$ by Lemma 3.2. Because $\text{id}_{S^op}U = n$, we have $\text{id}_{S^*}E \leq n$ by [9, Lemma 2.6(1)]. Because $\text{id}_RU = n$, we have $\text{pd}_{S^*}E \leq n$ by [9, Theorem 2.11]. If $\text{pd}_{S^*}E = m < n$, then by [9, Theorem 2.9], there exists an exact sequence

$$0 \to U_m \to \cdots \to U_1 \to U_0 \to E \to 0$$

5
in Mod $R$ with all $U_i$ in Add$_R U$, where Add$_R U$ is the full subcategory of Mod $R$ consisting of modules isomorphic to direct summands of direct sums of copies of $rU$. So Ext$_R^n(E,U) \cong \text{Ext}_{R}^{n-m}(U_m,U) = 0$ by [9, Proposition 2.2(2)]. It is a contradiction. Thus we have pd$_S^*E = n$. 

Lemma 3.4. The following statements are equivalent.

(1) There exists $0 \neq M \in \text{mod } R$ such that Ext$_R^{\geq 0}(M,U) = 0$.

(2) There exists an exact sequence

$$0 \to Q_0 \xrightarrow{k_1} Q_1 \xrightarrow{k_2} Q_2 \to \cdots$$

in mod $S^{op}$ with all $Q_i$ in add $U_S$, such that Ext$_{S^{op}}^1(L_1,U) \neq 0$ and Ext$_{S^{op}}^i(L_i,U) = 0$ with $i \geq 2$, where $L_i = \text{Coker } k_i$ for any $i \geq 1$.

Proof. (2) $\Rightarrow$ (1) By (2), we get the following exact sequence

$$\cdots \to Q_2^* \xrightarrow{k_2^*} Q_1^* \xrightarrow{k_1^*} Q_0^* \to M \to 0$$

in mod $R$, where $M = \text{Coker } k_1^*$. Then $M \cong \text{Ext}_{S^{op}}^1(L_1,U) \neq 0$ by assumption. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & Q_0 & \xrightarrow{k_1} & Q_1 & \xrightarrow{k_2} & Q_2 & \to & \cdots \\
& & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & & \\
0 & \to & M^* & \xrightarrow{k_1^{**}} & Q_0^{**} & \xrightarrow{k_2^{**}} & Q_1^{**} & \to & \cdots
\end{array}$$

Because the upper row is exact by assumption, so is the lower row. It implies Ext$_R^{\geq 0}(M,U) = 0$.

(1) $\Rightarrow$ (2) Let $0 \neq M \in \text{mod } R$ with Ext$_R^{\geq 0}(M,U) = 0$ and let

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

be an exact sequence in mod $R$ with all $P_i$ projective. Then we get the following exact sequence

$$0 \to P_0^* \xrightarrow{d_0^*} P_1^* \xrightarrow{d_1^*} P_2^* \to \cdots$$

(3.1) in mod $S^{op}$ with all $P_i^*$ in add $U_S$. Notice that $P_i \cong P_i^{**}$ naturally for any $i \geq 0$, so the sequence

$$\cdots \to P_2^{**} \xrightarrow{d_2^{**}} P_1^{**} \xrightarrow{d_1^{**}} P_0^{**}$$

in mod $R$ is exact. Set $L_i := \text{Coker } d_i^*$ for any $i \geq 1$. Then Ext$_{S^{op}}^1(L_1,U) \cong M \neq 0$ and Ext$_{S^{op}}^i(L_i,U) = 0$ for any $i \geq 2$, and so (3.1) is the desired exact sequence.
Following [4], an injective resolution

\[ 0 \to N \xrightarrow{ε_0} I_0 \xrightarrow{ε_1} I_1 \xrightarrow{ε_2} I_2 \to \cdots \]

of a module \( N \) in \( \text{Mod} S^{\text{op}} \) is said to have a redundant image if some \( \text{Im} ε_n = \oplus_{j=1}^m W_j \) such that each \( W_j \) is isomorphic to a direct summand of some \( \text{Im} ε_i \) with \( i_j \neq n \). It is clear that the minimal injective resolution of \( N_S \) has a redundant image if \( \text{id}_{S^{\text{op}}} N < \infty \).

Lemma 3.5. If \( U_S \) has an injective resolution with a redundant image, then \( M = 0 \) for any \( M \in \text{mod} R \) with \( \text{Ext}_R^{2}(M, U) = 0 \).

Proof. Let \( M \in \text{mod} R \) with \( \text{Ext}_R^0(M, U) = 0 \). If \( M \neq 0 \), then by Lemma 3.4 and its proof, we get an exact sequence

\[ 0 \to Q_0 \xrightarrow{k_1} Q_1 \xrightarrow{k_2} Q_3 \to \cdots \]

in \( \text{mod} S^{\text{op}} \) with all \( \text{Q}_i \) in \( \text{add} U_S \) such that \( \text{Ext}_{S^{\text{op}}}^1(L_1, U) \cong \text{Ext}_{S^{\text{op}}}^1(L_n, U) = 0 \) and \( \text{Ext}_{S^{\text{op}}}^1(L_1, U) = 0 \) for any \( i \geq 2 \).

Because \( U_S \) has an injective resolution with a redundant image, there exists an injective resolution

\[ 0 \to U_S \xrightarrow{α_n} I_0 \xrightarrow{α_1} I_1 \xrightarrow{α_2} I_2 \to \cdots \]

of \( U_S \) in \( \text{Mod} S^{\text{op}} \) with some \( \text{Im} α_n = \oplus_{j=1}^m W_j \) such that each \( W_j \) is isomorphic to a direct summand of some \( \text{Im} α_i \) with \( i_j \neq n \). Then we have

\[ \text{Ext}_{S^{\text{op}}}^1(L_1, U) \cong \text{Ext}_{S^{\text{op}}}^1(L_n, U) \cong \text{Ext}_{S^{\text{op}}}^1(L_n, \text{Im} α_n) \cong \oplus_{j=1}^m \text{Ext}_{S^{\text{op}}}^1(L_n, W_j). \]

Since

\[ \text{Ext}_{S^{\text{op}}}^1(L_n, \text{Im} α_{i_j}) \cong \text{Ext}_{S^{\text{op}}}^{i_j+1}(L_n, U) = 0 \]

by the above argument, we have \( \text{Ext}_{S^{\text{op}}}^1(L_n, W_j) = 0 \) for any \( 1 \leq j \leq m \). It follows that \( \text{Ext}_{S^{\text{op}}}^1(L_1, U) = 0 \), a contradiction. Consequently we conclude that \( M = 0 \).

We introduce the notion of Gorenstein modules as follows.

Definition 3.6. We called \( R U \) (resp. \( U_S \)) Gorenstein if \( U \)-lim.dim \( E_i \leq i \) (resp. \( U \)-lim.dim \( S_{S^{\text{op}}} \) \( E'_i \leq i \)) for any \( i \geq 0 \).

By Theorem 2.3, we have that \( R U \) is Gorenstein if and only if \( U_S \) is Gorenstein, and if and only if \( R U \) (equivalent \( U_S \)) is \( n \)-Gorenstein for all \( n \). Following Theorem 2.3 and [16, Theorem 3.6], we know that the notion defined above is a non-commutative version of that of Gorenstein modules in [16]. The following proposition is useful in proving the main result.
**Proposition 3.7.** Let $RU$ be a Gorenstein module with $\text{id}_RU = \text{id}_{S^{op}}U = n$. Then $\text{Ext}_S^n(N, U) \in \text{mod} \ R$ is Artinian for any $N \in \text{mod} \ S^{op}$.

**Proof.** Let $X \subseteq Y$ be left $R$-submodules of $\text{Ext}_S^n(N, U)$. Then by Theorem 2.3, we have

$$\text{Ext}_R^{0 \leq i < n}(X, U) = 0 = \text{Ext}_R^{0 \leq i < n}(Y, U).$$

So from the exact sequence

$$0 \to X \to Y \to Y/X \to 0,$$

we get $\text{Ext}_R^{0 \leq i < n}(Y/X, U) = 0$ and the following exact sequence

$$0 \to \text{Ext}_R^n(Y/X, U) \to \text{Ext}_R^n(Y, U) \to \text{Ext}_R^n(X, U) \to 0.$$

Put $A_0 := \text{Ext}_S^n(N, U)$ and let

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

be a descending chain of left $R$-submodules of $A_0$. Then from the commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \to & A_{i+1} & \to & A_0 & \to & A/A_{i+1} & \to & 0 \\
0 & \to & A_i & \to & A_0 & \to & A/A_i & \to & 0,
\end{array}$$

we get the following commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \to & \text{Ext}_R^n(A_0/A_i, U) & \to & \text{Ext}_R^n(A_0, U) & \to & \text{Ext}_R^n(A_i, U) & \to & 0 \\
0 & \to & \text{Ext}_R^n(A_0/A_{i+1}, U) & \to & \text{Ext}_R^n(A_0, U) & \to & \text{Ext}_R^n(A_{i+1}, U) & \to & 0.
\end{array}$$

We can regard $\text{Ext}_R^n(A_0/A_i, U)$ as a right $S$-submodule of $\text{Ext}_R^n(A_0, U)$. Then the following sequence is the ascending chain of right $S$-submodules of $\text{Ext}_R^n(A_0, U)$:

$$\text{Ext}_R^n(A_0/A_1, U) \subseteq \text{Ext}_R^n(A_0/A_2, U) \subseteq \cdots \subseteq \text{Ext}_R^n(A_0, U). \quad (3.2)$$

Since $A_0 \in \text{mod} \ R$, $\text{Ext}_R^n(A_0, U) \in \text{mod} \ S^{op}$ is Noetherian. So (3.2) terminates at some step $m$. Then from the exact sequence

$$0 \to A_m/A_{m+1} \to A_0/A_{m+1} \to A_0/A_m \to 0,$$

we get the following exact sequence

$$\text{Ext}_R^n(A_0/A_m, U) \xrightarrow{\cong} \text{Ext}_R^n(A_0/A_{m+1}, U) \to \text{Ext}_R^n(A_m/A_{m+1}, U) \to 0.$$
and $\text{Ext}^n_R(A_m/A_{m+1}, U) = 0$. Then by the above argument, we have that $\text{Ext}^n_R(A_m/A_{m+1}, U) = 0$. Because $\text{id}_R U = n$, we have $\text{Ext}^n_R(A_m/A_{m+1}, U) = 0$. Because $\text{id}_{S^n} U = n$, by Lemma 3.5 we have that $A_m/A_{m+1} = 0$ and $A_m = A_{m+1}$. Thus $A_0(= \text{Ext}^n_{S^n}(N, U))$ is an Artinian left $R$-module.

By Proposition 3.7, we get the following

**Corollary 3.8.** Let $R U$ be a Gorenstein module with $\text{id}_R U = \text{id}_{S^n} U = n$ and $N \in \text{mod} S^{op}$. Then we have

1. $\text{Ext}^n_{S^n}(N, U)$ embeds in $E^{(t)}_n$ for some $t \geq 1$.

2. If $M$ is a non-zero $R$-submodule of $\text{Ext}^n_{S^n}(N, U)$, then $\text{Ext}^n_R(M, U) \neq 0$.

**Proof.** (1) Let $N \in \text{mod} S^{op}$. Then $\text{Ext}^n_{S^n}(N, U) \in \text{mod} R$ embeds in a finite direct sum of copies of $\bigoplus_{i=0}^{n-1} E_i$ by [8, Lemma 17.2.5]. On the other hand, by [3, Proposition VI.5.3] and Theorem 2.4, we have

$$\text{Hom}_R(\text{Ext}^n_{S^n}(N, U), E_i) \cong \text{Tor}^n_R(N, * E_i) = 0$$

for any $0 \leq i \leq n - 1$. So there exists some $t \geq 1$ such that $\text{Ext}^n_{S^n}(N, U)$ embeds in $E^{(t)}_n$.

(2) Let $M$ be a non-zero $R$-submodule of $\text{Ext}^n_{S^n}(N, U)$. Because we have the following exact sequence

$$0 \rightarrow K^{(t)}_{n-1} \rightarrow E^{(t)}_{n-1} \rightarrow E^{(t)}_n \rightarrow 0$$

in $\text{Mod} R$ with $K^{(t)}_{n-1} \rightarrow E^{(t)}_{n-1}$ an essential monomorphism, by (1) and Lemma 3.1 we have that $\text{Ext}^n_R(M, U)^{(t)} \cong \text{Ext}^1_R(M, K^{(t)}_{n-1}) \cong \text{Ext}^1_R(M, K^{(t)}_{n}) \neq 0$ and $\text{Ext}^n_R(M, U) \neq 0$. □

We now are in a position to prove the following

**Theorem 3.9.** Let $R U$ be a Gorenstein module with $\text{id}_R U = \text{id}_{S^n} U = n$. Then any non-zero submodule of $E_n$ has a non-zero Artinian submodule.

**Proof.** Let $V$ be a non-zero submodule of $E_n$ and $E$ a non-zero indecomposable direct summand of the injective envelope $Q$ of $V$. Then $E$ is the injective envelope of some finitely generated non-zero $R$-submodule $X$ of $E$ by [14, Theorem 2.4]. By Lemma 3.2, we have $\text{Ext}^n_R(X, U) \neq 0$. Let $I$ be an injective cogenerator for $\text{Mod} S^{op}$. Then by [3, Proposition VI.5.3], we have

$$\text{Tor}^n_R(\ast I, X) \cong \text{Hom}_{S^{op}}(\text{Ext}^n_R(X, U), I) \neq 0.$$ 

Because $\text{id}_{S^{op}} \ast I = \text{id}_R U = n$ by [9, Lemma 2.6(2)] and assumption, the inclusion $X \hookrightarrow E$ induces a monomorphism $\text{Tor}_n^R(\ast I, X) \hookrightarrow \text{Tor}_n^R(\ast I, E)$. It yields $\text{Tor}_n^R(\ast I, E) \neq 0$. By [11, Lemmas 5.1(c) and
4.1], we have that \( \text{Tor}^R_{\leq 1}(I, U) = 0 \) and both \( I \) and \( E \) are in the Bass class with respect to \( RUS \). So we get the following isomorphisms

\[
\begin{align*}
\text{Tor}^S_n(I, \ast E) & \cong \text{Tor}^S_n(I \otimes_R U, \ast E) \quad \text{(by [11, Lemma 4.1])} \\
& \cong \text{Tor}^R_n(\ast I, E) \quad \text{(by [11, Theorem 6.4(c)])},
\end{align*}
\]

and hence \( \text{Tor}^S_n(I, \ast E) \neq 0 \). Let \( \{N_i\} \) be the set of all finitely generated right \( S \)-submodules of \( I \). Because \( I = \lim_{\to} N_i \) and the functor Tor commutes with \( \lim_{\to} \) by [15, Example 5.32(iii) and Proposition 7.8], there exists some \( N_i \) such that \( \text{Tor}^S_n(N_i, \ast E) \neq 0 \). Then by [3, Proposition VI.5.3] again, we have

\[
\text{Hom}_R(\text{Ext}^{\ast}S_{\text{op}}(N_i, U), E) \cong \text{Tor}^S_n(N_i, \ast E) \neq 0.
\]

So there exists a non-zero homomorphism \( f : \text{Ext}^{\ast}S_{\text{op}}(N_i, U) \to E \) in \( \text{Mod} R \). By Proposition 3.7, \( \text{Ext}^{\ast}S_{\text{op}}(N_i, U) \in \text{mod} R \) is Artinian. Thus as an \( R \)-quotient module of \( \text{Ext}^{\ast}S_{\text{op}}(N_i, U) \), \( \text{Im } f \) is a non-zero Artinian \( R \)-submodule of \( E(< Q) \). Because \( V \) is essential in \( Q \), we have that \( V \cap \text{Im } f \) is a non-zero Artinian \( R \)-submodule of \( V \).

Theorem 1.2 is a special case of the following result.

**Corollary 3.10.** Let \( R\U \) be a Gorenstein module with \( \text{id}_R U = \text{id}_{S_{\text{op}}} U = n \). Then any non-zero \( R \)-submodule of \( E_n \) has an essential socle.

**Proof.** By Theorem 3.9, we have that any non-zero \( R \)-submodule of \( E_n \) has a non-zero Artinian \( R \)-submodule. Now the assertion follows from [1, Corollary 9.10].

The second assertion in the following result is a supplement to Proposition 3.3(1).

**Corollary 3.11.** Let \( R\U \) be a Gorenstein module with \( \text{id}_R U = \text{id}_{S_{\text{op}}} U = n \). Then we have

1. Any non-zero direct summand of \( E_n \) is a direct sum of the injective envelopes of some simple left \( R \)-modules.

2. If \( n \geq 1 \) and \( S \) is further a right Artinian ring, then \( \oplus_{i=0}^{n-1} E_i \) and \( E_n \) have no isomorphic non-zero direct summands.

**Proof.** (1) By Corollary 3.10 and [14, Proposition 2.1].

(2) If \( S \) is a right Artinian ring, then \( \text{pd}_S(\oplus_{i=0}^{n-1} E_i) = \text{fd}_S(\oplus_{i=0}^{n-1} E_i) \leq n - 1 \) by Theorem 2.3. So \( \text{pd}_S(\ast E) \leq n - 1 \) for any direct summand \( E' \) of \( \oplus_{i=0}^{n-1} E_i \). On the other hand, we have \( \text{pd}_S(\ast E) = n \) for any non-zero direct summand \( E \) of \( E_n \) by Proposition 3.3(2). Thus the assertion follows.
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References


