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# ON VARIOUS RESTRICTED SUMSETS

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Abstract. For finite subsets $A_1, \ldots, A_n$ of a field, their sumset is given by $\{a_1 + \cdots + a_n : a_1 \in A_1, \ldots, a_n \in A_n\}$. In this paper we study various restricted sumsets of $A_1, \ldots, A_n$ with restrictions of the following forms:

$$a_i - a_j \notin S_{ij}, \text{ or } \alpha_i a_i \neq \alpha_j a_j, \text{ or } a_i + b_i \not\equiv a_j + b_j \pmod{m_{ij}}.$$

Furthermore, we gain an insight into relations among recent results on this area obtained in quite different ways.

## 1. Introduction

The additive order of the identity of a field $F$ is either infinite or a prime, we call it the *characteristic* of $F$.

Let $F$ be a field of characteristic $p$, and let $A_1, \ldots, A_n$ be finite subsets of $F$ with $0 < k_1 = |A_1| \leqslant \cdots \leqslant k_n = |A_n|$. Concerning various restricted sumsets of $A_1, \ldots, A_n$, the following results are known:

(i) (The Cauchy-Davenport theorem (see, e.g. [N]))

$$|\{a_1 + \cdots + a_n : a_1 \in A_1, \ldots, a_n \in A_n\}| \geqslant \min\{p, k_1 + \cdots + k_n - n + 1\}.$$

(ii) (Dias da Silva and Hamidoune [DH]) If $A_1 = \cdots = A_n = A$, then

$$|\{a_1 + \cdots + a_n : a_i \in A,\ a_1, \ldots, a_n \text{ are distinct}\}| \geqslant \min\{p, n|A| - n^2 + 1\}.$$

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(iii) (Alon, Nathanson and Ruzsa [ANR2]) If $k_1 < \cdots < k_n$, then

$$|\{a_1+\cdots+a_n : a_i \in A_i,\ a_i \neq a_j \text{ if } i \neq j\}| \geqslant \min\left\{p, \sum_{i=1}^n k_i - \frac{n(n+1)}{2} + 1\right\}.$$

(iv) (Hou and Sun [HS]) Let $S_{ij}$ $(1 \leqslant i, j \leqslant n,\ i \neq j)$ be finite subsets of $F$ with cardinality $m$. If $k_1 = \cdots = k_n = k$ and $p > \max\{ln, mn\}$ where $l = k - 1 - m(n-1)$, then

$$|\{a_1 + \cdots + a_n : a_i \in A_i,\ a_i - a_j \notin S_{ij} \text{ if } i \neq j\}| \geqslant ln + 1.$$

(v) (Liu and Sun [LS]) Let $P_1(x), \ldots, P_n(x) \in F[x]$ be monic and of degree $m > 0$. If $k_n > m(n-1)$, $k_{i+1} - k_i \in \{0, 1\}$ for all $i = 1, \ldots, n-1$, and $p > K = (k_n - 1)n - (m+1)\binom{n}{2}$, then we have

$$|\{a_1 + \cdots + a_n : a_i \in A_i,\ P_i(a_i) \neq P_j(a_j) \text{ if } i \neq j\}| \geqslant K + 1.$$

(vi) (Sun [Su]) Let $P_1(x), \ldots, P_n(x) \in F[x]$ have degree $m > 0$ with the permanent of the matrix $(b_j^{i-1})_{1\leqslant i,j\leqslant n}$ nonzero, where $b_j$ is the leading coefficient of $P_j(x)$. If $k_1 = \cdots = k_n = k > m(n-1)$ and $K = (k-1)n - (m+1)\binom{n}{2} < p$, then

$$|\{a_1 + \cdots + a_n : a_i \in A_i,\ a_i \neq a_j \ \&\ P_i(a_i) \neq P_j(a_j) \text{ if } i \neq j\}| \geqslant K + 1.$$

While result (ii) was deduced by a deep tool from the representation theory of symmetric groups, results (iii)–(vi) were obtained from the following basic principle arising from Alon and Tarsi [AT].

**Combinatorial Nullstellensatz** ([A1, A3]). *Let $A_1, \ldots, A_n$ be finite subsets of a field $F$ with $|A_i| > k_i$ for $i = 1, \ldots, n$ where $k_1, \ldots, k_n$ are nonnegative integers. If the coefficient of the monomial $x_1^{k_1} \cdots x_n^{k_n}$ in $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ is nonzero and $k_1+\cdots+k_n$ is the total degree of $f$, then there are $a_1 \in A_1, \ldots, a_n \in A_n$ such that $f(a_1, \ldots, a_n) \neq 0$.*

Lower bounds for various restricted sumsets are usually yielded with help of the following lemma (or Proposition 2.1 of [HS]) implied by the Combinatorial Nullstellensatz.

**Lemma 1.1** (Alon et al. [ANR1, ANR2]). *Let $A_1, \ldots, A_n$ be finite nonempty subsets of a field $F$ with $k_i = |A_i|$ for $i = 1, \ldots, n$. Let $P(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n] \setminus \{0\}$ and $\deg P \leqslant \sum_{i=1}^n (k_i - 1)$. If the coefficient of the monomial $x_1^{k_1-1} \cdots x_n^{k_n-1}$ in the polynomial*

$$P(x_1, \ldots, x_n)(x_1 + \cdots + x_n)^{\sum_{i=1}^n (k_i-1) - \deg P}$$

*does not vanish, then we have*

$$|\{a_1+\cdots+a_n : a_i\in A_i,\ P(a_1,\ldots,a_n)\neq 0\}| \geqslant \sum_{i=1}^{n}(k_i-1)-\deg P+1.$$

In the next section, we will develop a general technique to compute certain coefficients of some polynomials. Using Lemma 1.1 and our work in Section 2, we will prove the following main theorems in Section 3.

**Theorem 1.1.** *Let $F$ be a field of characteristic $p$, and let $A_1,\ldots,A_n$ be finite nonempty subsets of $F$ with $|A_n|=k$ and $|A_{i+1}|=|A_i|+1$ for $i=1,\ldots,n-1$. Let $m$ be a positive integer, and let $S_{ij}\subseteq F$ and $|S_{ij}|<2m$ for all $1\leqslant i<j\leqslant n$. If $p>\max\{mn,(k-1)n-mn(n-1)\}$, then we have*

$$|\{a_1+\cdots+a_n : a_i\in A_i,\ \text{and}\ a_i-a_j\notin S_{ij}\ \text{if}\ i<j\}| \geqslant (k-1-m(n-1))n+1.$$

*Remark* 1.1. Theorem 1.1 can be viewed as a partial generalization of result (iii).

**Theorem 1.2.** *Let $k$ and $m$ be positive integers. Let $A_1,\ldots,A_n$ be subsets of the complex field $\mathbb{C}$ with cardinality $k$, and let $S_{ij}$ $(1\leqslant i<j\leqslant n)$ be subsets of $\mathbb{C}$ with at most $2m-1$ elements. If $\zeta_1,\ldots,\zeta_n$ are distinct $q$th roots of unity where $q$ is a positive odd integer, then*

$$\left|\left\{\sum_{i=1}^{n}a_i : a_i\in A_i,\ a_i\zeta_i\neq a_j\zeta_j\ \text{and}\ a_i-a_j\notin S_{ij}\ \text{if}\ i<j\right\}\right| \geqslant (k-1-m(n-1))n+1.$$

*Remark* 1.2. A conjecture of Snevily [S] states that for any cyclic group with odd order if $A$ and $B=\{b_1,\ldots,b_n\}$ are its subsets with cardinality $n$ then there is a numbering $\{a_i\}_{i=1}^n$ of the elements of $A$ such that $a_1b_1,\ldots,a_nb_n$ are pairwise distinct. Using the Combinatorial Nullstellensatz Alon [A2] confirmed this for the cyclic group $\mathbb{Z}/p\mathbb{Z}$ where $p$ is an odd prime. Since we can identify a cyclic group of order $q$ with the multiplicative group of all the $q$th roots of unity, Snevily's conjecture follows from Theorem 1.2 in the case $k=n$, $m=1$, $A_1=\cdots=A_n=A$ and $S_{ij}=\{0\}$ $(1\leqslant i<j\leqslant k)$, which was first obtained by Dasgupta, Károlyi, Serra and Szegedy [DKSS] in 2001. Another extension of Snevily's conjecture appeared in [Su].

**Theorem 1.3.** *Let $\alpha_1,\ldots,\alpha_n$ be positive reals, and let $b_1,\ldots,b_n$ be integers. Let $A_1,\ldots,A_n$ be finite subsets of $\mathbb{Z}$ with cardinality $k>0$. For $1\leqslant i<j\leqslant n$ let $m_{ij}$ be an integer greater than $2\max\{|x_i-x_j|:\ x_i\in A_i,\ x_j\in A_j\}|$. Then the restricted sumset*

$$\left\{\sum_{i=1}^n a_i:\ a_i\in A_i,\ a_i\alpha_i\neq a_j\alpha_j \text{ and } a_i+b_i\not\equiv a_j+b_j \pmod{m_{ij}} \text{ if } i<j\right\}$$

*has more than $(k-n)n$ elements.*

**Corollary 1.1** (Kézdy and Snevily [KS])**.** *Let $m$ and $n$ be positive integers with $n\leqslant(m+1)/2$. Then, for any $b_1,\ldots,b_n\in\mathbb{Z}$, there exists a permutation $\sigma$ on $\{1,\ldots,n\}$ such that $1+b_{\sigma(1)},\ldots,n+b_{\sigma(n)}$ are pairwise distinct modulo $m$.*

*Proof.* Observe that $m/2>n-1=\max_{1\leqslant i\leqslant j\leqslant n}(j-i)$. Applying Theorem 1.3 with $\alpha_1=\cdots=\alpha_n=1$ and $A_1=\cdots=A_n=\{1,\ldots,n\}$, we find that there exists a permutation $\tau$ on $\{1,\ldots,n\}$ such that $\tau(1)+b_1,\ldots,\tau(n)+b_n$ are pairwise distinct modulo $m$. So the desired result follows. □

*Remark* 1.3. In [KS] Corollary 1.1 was applied to tree embeddings. Snevily [S] even conjectured that the condition $n\leqslant(m+1)/2$ in Corollary 1.1 can be weakened by $n<m$.

Let $G=\{a_1,\ldots,a_n\}$ be an additive abelian group of order $n$, and let $b_1,\ldots,b_n$ be elements of $G$ with $b_1+\cdots+b_n=0$. In 1952 M. Hall [H] proved that there exists a permutation $\sigma$ on $\{1,\ldots,n\}$ such that $a_1+b_{\sigma(1)},\ldots,a_n+b_{\sigma(n)}$ are pairwise distinct. Let $\sigma$ be a permutation on $\{1,\ldots,n\}$ such that $b_1-a_{\sigma(1)},\ldots,b_n-a_{\sigma(n)}$ are pairwise distinct. Assume that $n>1$ and $a_n=b_n=0$. Then there exists a permutation $\sigma'$ on $\{1,\ldots,n-1\}$ such that

$$a_{\sigma'(i)}=a_{\sigma(i)}-a_{\sigma(n)}\neq 0 \quad \text{for every } i=1,\ldots,n-1.$$

Since $\{b_i-a_{\sigma'(i)}:\ i=1,\ldots,n-1\}=G\setminus\{0\}$, there is a permutation $\tau$ on $\{1,\ldots,n-1\}$ such that for any $i=1,\ldots,n-1$ we have $b_i-a_{\sigma'(i)}=a_{\tau(i)}$ and hence $b_i=a_{\sigma'(i)}+a_{\tau(i)}$. In the case $G=\mathbb{Z}/n\mathbb{Z}$, this provides a positive answer to an open question of Parker (cf. [G]).

## 2. Relations among coefficients of certain polynomials

As usual, we let $(x)_0=1$ and $(x)_n=x(x-1)\cdots(x-n+1)$ for $n=1,2,3,\ldots$. For a polynomial

$$P(x_1,\ldots,x_n)=\sum_{j_1,\ldots,j_n} a_{j_1,\ldots,j_n}x_1^{j_1}\cdots x_n^{j_n}$$

over a commutative ring, we write $[x_1^{j_1}\cdots x_n^{j_n}]P(x_1,\dots,x_n)$ to denote the coefficient $a_{j_1,\dots,j_n}$.

**Lemma 2.1.** *Let*

$$P(x_1,\dots,x_n)=\sum_{\substack{j_1,\dots,j_n\geqslant 0\\ j_1+\cdots+j_n=m}} c_{j_1,\dots,j_n}x_1^{j_1}\cdots x_n^{j_n}\in\mathbb{C}[x_1,\dots,x_n]$$

*and*

$$P^*(x_1,\dots,x_n)=\sum_{\substack{j_1,\dots,j_n\geqslant 0\\ j_1+\cdots+j_n=m}} c_{j_1,\dots,j_n}(x_1)_{j_1}\cdots(x_n)_{j_n}.$$

*Suppose that* $0\leqslant\deg P\leqslant k_1+\cdots+k_n$ *where* $k_1,\dots,k_n$ *are nonnegative integers. Then*

$$[x_1^{k_1}\cdots x_n^{k_n}]P(x_1,\dots,x_n)(x_1+\cdots+x_n)^{k_1+\cdots+k_n-\deg P}$$

*coincides with*

$$\frac{(\sum_{i=1}^n k_i-\deg P)!}{k_1!\cdots k_n!}P^*(k_1,\dots,k_n).$$

*Proof.* Let $K=k_1+\cdots+k_n-\deg P$. Then

$$\begin{aligned}&[x_1^{k_1}\cdots x_n^{k_n}]P(x_1,\dots,x_n)(x_1+\cdots+x_n)^K\\ =&[x_1^{k_1}\cdots x_n^{k_n}]\sum_{\substack{j_1,\dots,j_n\geqslant 0\\ j_1+\cdots+j_n=m}} c_{j_1,\dots,j_n}x_1^{j_1}\cdots x_n^{j_n}\sum_{\substack{i_1,\dots,i_n\geqslant 0\\ i_1+\cdots+i_n=K}} K!\frac{x_1^{i_1}\cdots x_n^{i_n}}{i_1!\cdots i_n!}\\ =&K!\sum_{\substack{j_1,\dots,j_n\geqslant 0\\ j_1+\cdots+j_n=m}} c_{j_1,\dots,j_n}\frac{(k_1)_{j_1}\cdots(k_n)_{j_n}}{k_1!\cdots k_n!}\\ =&\frac{K!}{k_1!\cdots k_n!}P^*(k_1,\dots,k_n).\end{aligned}$$

This concludes the proof. □

Let $S_n$ denote the symmetric group of all permutations on $\{1,\dots,n\}$. For $\sigma\in S_n$ we let $\varepsilon(\sigma)$ be 1 or $-1$ according to whether $\sigma$ is even or odd. For a matrix $A=(a_{ij})_{1\leqslant i,j\leqslant n}$ over a field the determinant and the permanent of $A$ are defined by

$$\|A\|=\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n a_{i,\sigma(i)}\quad\text{and}\quad \mathrm{per}(A)=\sum_{\sigma\in S_n}\prod_{i=1}^n a_{i,\sigma(i)}$$

respectively.

Lemma 2.1 is very useful. For example, in view of Lemmas 1.1 and 2.1, result (iii) follows from the following simple observation:

$$\begin{aligned}\|x_j^{i-1}\|^*_{1\leqslant i,j\leqslant n} =& \sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n (x_{\sigma(i)})_{i-1} \\ =&\|(x_j)_{i-1}\|_{1\leqslant i,j\leqslant n} = \|x_j^{i-1}\|_{1\leqslant i,j\leqslant n},\end{aligned} \tag{2.1}$$

where in the last step we note that $x^r$ $(0\leqslant r<n)$ can be written as a linear combination of $(x)_0,\ldots,(x)_r$.

Now we present our main technique concerning the operator $P\mapsto P^*$.

**Theorem 2.1.** *Let $m_1,\ldots,m_n$ be nonnegative integers, and let $A=(a_{ij})_{1\leqslant i,j\leqslant n}$ be a matrix over $\mathbb{C}$. Set*

$$f(x_1,\ldots,x_n)=\|a_{ij}x_j^{m_i}\|_{1\leqslant i,j\leqslant n}P(x_1,\ldots,x_n),$$

*where $P(x_1,\ldots,x_n)\in\mathbb{C}[x_1,\ldots,x_n]$ is homogeneous and*

$$P(x_1,\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_{j-1},x_i,x_{j+1},\ldots,x_n)=\nu P(x_1,\ldots,x_n)$$

*for all $1\leqslant i<j\leqslant n$ with a fixed $\nu\in\{1,-1\}$. Then*

$$f^*(x,\ldots,x)=P^*(x-m_1,\ldots,x-m_n)\prod_{i=1}^n (x)_{m_i}\times\begin{cases}\|A\| & \text{if } \nu=1,\\ \operatorname{per}(A) & \text{if } \nu=-1.\end{cases}$$

*Proof.* Any $\sigma\in S_n$ can be written as a product of transpositions:

$$\sigma=(i_1j_1)\cdots(i_rj_r)\quad\text{where } 1\leqslant i_s<j_s\leqslant n \text{ for } s=1,\ldots,r.$$

Thus

$$P(x_{\sigma(1)},\ldots,x_{\sigma(n)})=\nu^rP(x_1,\ldots,x_n)=\varepsilon(\sigma)^{(1-\nu)/2}P(x_1,\ldots,x_n).$$

Write $P(x_1,\ldots,x_n)=\sum_{j_1,\ldots,j_n}c_{j_1,\ldots,j_n}x_1^{j_1}\cdots x_n^{j_n}$. Then

$$\begin{aligned}&f(x_1,\ldots,x_n)\\ =&\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n\left(a_{i,\sigma(i)}x_{\sigma(i)}^{m_i}\right)\times P(x_1,\ldots,x_n)\\ =&\sum_{\sigma\in S_n}\left(\varepsilon(\sigma)\prod_{i=1}^n\left(a_{i,\sigma(i)}x_{\sigma(i)}^{m_i}\right)\times\varepsilon(\sigma)^{(\nu-1)/2}P(x_{\sigma(1)},\ldots,x_{\sigma(n)})\right)\\ =&\sum_{\sigma\in S_n}\left(\varepsilon(\sigma)^{(\nu+1)/2}\prod_{i=1}^n a_{i,\sigma(i)}\times\sum_{j_1,\ldots,j_n}c_{j_1,\ldots,j_n}\prod_{i=1}^n x_{\sigma(i)}^{m_i+j_i}\right).\end{aligned}$$

Therefore

$$\begin{aligned} f^*(x,\dots,x) =& \sum_{\sigma\in S_n} \varepsilon(\sigma)^{(\nu+1)/2}\prod_{i=1}^n a_{i,\sigma(i)} \times \sum_{j_1,\dots,j_n} c_{j_1,\dots,j_n}\prod_{i=1}^n (x)_{m_i+j_i} \\ =& a\sum_{j_1,\dots,j_n} c_{j_1,\dots,j_n}\left(\prod_{i=1}^n (x)_{m_i}\times\prod_{i=1}^n (x-m_i)_{j_i}\right) \\ =& a\prod_{i=1}^n (x)_{m_i}\times P^*(x-m_1,\dots,x-m_n), \end{aligned}$$

where

$$a=\sum_{\sigma\in S_n}\varepsilon(\sigma)^{(\nu+1)/2}\prod_{i=1}^n a_{i,\sigma(i)}=\begin{cases} \|A\| & \text{if } \nu=1, \\ \mathrm{per}(A) & \text{if } \nu=-1. \end{cases}$$

This concludes the proof. □

**Corollary 2.1.** *Let $m_1,\dots,m_n$ be nonnegative integers.*

(i) (Sun [Su]) *If $A=(a_{ij})_{1\leqslant i,j\leqslant n}$ is a matrix with $a_{ij}\in\mathbb{C}$, then*

$$\begin{aligned} &\left(\|a_{ij}x_j^{m_i}\|_{1\leqslant i,j\leqslant n}\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^\delta\right)^*(x,\dots,x) \\ =&\prod_{1\leqslant i<j\leqslant n}(m_i-m_j)^\delta\times\prod_{i=1}^n (x)_{m_i}\times\begin{cases} \|A\| & \text{if } \delta=0, \\ \mathrm{per}(A) & \text{if } \delta=1. \end{cases} \end{aligned} \tag{2.2}$$

(ii) We have

$$\|x_j^{m_i}\|^*_{1\leqslant i,j\leqslant n}(x-n+1,\dots,x)=\prod_{1\leqslant i<j\leqslant n}(m_j-m_i)\times\prod_{i=1}^n\frac{(x)_{m_i}}{(x)_{i-1}}. \tag{2.3}$$

*Proof.* As $\|x_j^{i-1}\|_{1\leqslant i,j\leqslant n}=\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)$ (Vandermonde), by (2.1) we have

$$\begin{aligned} &\left(\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^\delta\right)^*(x-m_1,\dots,x-m_n) \\ =&\prod_{1\leqslant i<j\leqslant n}(x-m_j-(x-m_i))^\delta=\prod_{1\leqslant i<j\leqslant n}(m_i-m_j)^\delta. \end{aligned}$$

In view of this, Theorem 2.1 yields (2.2) immediately.

By Theorem 2.1,

$$\begin{aligned}&\left(\|x_j^{n-i}\|_{1\leqslant i,j\leqslant n}\times\|x_j^{m_i}\|_{1\leqslant i,j\leqslant n}\right)^*(x,\ldots,x)\\=&n!\prod_{i=1}^n(x)_{n-i}\times\|x_j^{m_i}\|^*_{1\leqslant i,j\leqslant n}(x-n+1,\ldots,x).\end{aligned}$$

On the other hand, by part (i) we have

$$\begin{aligned}&\left(\|x_j^{n-i}\|_{1\leqslant i,j\leqslant n}\times\|x_j^{m_i}\|_{1\leqslant i,j\leqslant n}\right)^*(x,\ldots,x)\\=&(-1)^{\binom n2}\left(\|x_j^{m_i}\|_{1\leqslant i,j\leqslant n}\times\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)\right)^*(x,\ldots,x)\\=&n!\prod_{1\leqslant i<j\leqslant n}(m_j-m_i)\times\prod_{i=1}^n(x)_{m_i}.\end{aligned}$$

So (2.3) follows.

The proof of Corollary 2.1 is now complete. □

*Remark* 2.1. When $m_i=(i-1)m$ for $i=1,\ldots,n$, Corollary 2.1(ii) yields the following result related to [LS]:

$$\begin{aligned}&\left(\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)\right)^*(x-n+1,\ldots,x)\\=&1!2!\cdots(n-1)!m^{n(n-1)/2}\frac{(x)_0(x)_m\cdots(x)_{(n-1)m}}{(x)_0(x)_1\cdots(x)_{n-1}}.\end{aligned}\tag{2.4}$$

**Theorem 2.2.** *Let $m$ be any positive integer, and let $a_1,\ldots,a_n$ be complex numbers. Then*

$$\begin{aligned}&\left(\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^{2m-1}\right)^*(x-n+1,\ldots,x)\\=&(-1)^{(m-1)\binom n2}\frac{m!(2m)!\cdots(nm)!}{(m!)^nn!}\times\frac{(x)_0(x)_m\cdots(x)_{(n-1)m}}{(x)_0(x)_1\cdots(x)_{n-1}}\end{aligned}\tag{2.5}$$

*and*

$$\begin{aligned}&\left(\prod_{1\leqslant i<j\leqslant n}(a_jx_j-a_ix_i)(x_j-x_i)^{2m-1}\right)^*(x,\ldots,x)\\=&(-1)^{m\binom n2}\frac{m!(2m)!\cdots(nm)!}{(m!)^nn!}\mathrm{per}(a_j^{i-1})_{1\leqslant i,j\leqslant n}\prod_{r=0}^{n-1}(x)_{rm}.\end{aligned}\tag{2.6}$$

*Proof.* Let $P_h(x_1,\dots,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^h$ for $h=1,2,3,\dots$. In light of Theorem 2.1,

$$\begin{aligned}P_{2m}^*(x,\dots,x)=&\left((-1)^{\binom n2}\|x_j^{n-i}\|_{1\leqslant i,j\leqslant n}P_{2m-1}(x_1,\dots,x_n)\right)^*(x,\dots,x)\\=&(-1)^{\binom n2}n!\prod_{i=1}^n(x)_{n-i}\times P_{2m-1}^*(x-n+1,\dots,x)\end{aligned}$$

and

$$\begin{aligned}&\left(\|a_j^{i-1}x_j^{i-1}\|_{1\leqslant i,j\leqslant n}P_{2m-1}(x_1,\dots,x_n)\right)^*(x,\dots,x)\\=&\frac{\mathrm{per}(a_j^{i-1})_{1\leqslant i,j\leqslant n}}{\mathrm{per}(a_j^0)_{1\leqslant i,j\leqslant n}}\left(\|a_j^0x_j^{i-1}\|_{1\leqslant i,j\leqslant n}P_{2m-1}(x_1,\dots,x_n)\right)^*(x,\dots,x)\\=&\frac{\mathrm{per}(a_j^{i-1})_{1\leqslant i,j\leqslant n}}{n!}P_{2m}^*(x,\dots,x).\end{aligned}$$

By Theorem 3.1 of Hou and Sun [HS],

$$P_{2m}^*(x,\dots,x)=(-1)^{m\binom n2}\frac{m!(2m)!\cdots(nm)!}{(m!)^n}(x)_0(x)_m\cdots(x)_{(n-1)m}.$$

So we have the desired (2.5) and (2.6). □

**Corollary 2.2.** *Let $k,m,n$ be positive integers with $k>m(n-1)$. Then*

$$\begin{aligned}&[x_1^{k-n}\cdots x_n^{k-1}](x_1+\cdots+x_n)^{(k-1)n-mn(n-1)}\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^{2m-1}\\&=(-1)^{(m-1)\binom n2}\frac{m!(2m)!\cdots(nm)!}{(m!)^nn!}\cdot\frac{((k-1-m(n-1))n)!}{\prod_{r=0}^{n-1}(k-1-rm)!}.\end{aligned}\tag{2.7}$$

*In particular,*

$$\left[\prod_{i=1}^n x_i^{(m-1)(n-1)+i-1}\right]\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^{2m-1}=(-1)^{(m-1)\binom n2}\frac{(mn)!}{(m!)^nn!}.\tag{2.8}$$

*Proof.* Combining Lemma 2.1 with (2.5) we obtain (2.7). Eq. (2.8) follows from (2.7) in the case $k=m(n-1)+1$. □

*Remark* 2.2. Let $m_1,\dots,m_n$ be nonnegative integers. A confirmed conjecture of Dyson [D] can be stated as follows:

$$\begin{aligned}&[x_1^{m_1(n-1)}\cdots x_n^{m_n(n-1)}]\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^{m_i+m_j}\\&=(-1)^{\sum_{j=1}^n(j-1)m_j}\frac{(m_1+\cdots+m_n)!}{m_1!\cdots m_n!}.\end{aligned}$$

(See, e.g., Zeilberger [Z].) Compared with this deep result, our (2.8) seems interesting too.

## 3. Proofs of Theorems 1.1–1.3

*Proof of Theorem 1.1.* The case $n=1$ or $k-1<m(n-1)$ is trivial. Below we assume $n\geqslant 2$ and $l=k-1-m(n-1)\geqslant 0$.

As $|F|\geqslant p>mn\geqslant 2m$ we can extend each $S_{ij}$ $(1\leqslant i<j\leqslant n)$ to a subset $S_{ij}^*$ of $F$ with cardinality $2m-1$. By Lemma 1.1 it suffices to show that

$$[x_1^{k-n}\cdots x_n^{k-1}](x_1+\cdots+x_n)^{ln}\prod_{1\leqslant i<j\leqslant n}\prod_{c\in S_{ij}^*}(x_j-x_i+c)$$

does not vanish. Let $e$ denote the multiplicative identity of the field $F$. Then the above coefficient equals $he$ where

$$h=[x_1^{k-n}\cdots x_n^{k-1}](x_1+\cdots+x_n)^{ln}\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)^{2m-1}\in\mathbb{Z}.$$

By Corollary 2.2,

$$h=(-1)^{(m-1)\binom n2}\frac{m!(2m)!\cdots(nm)!}{(m!)^n n!}\cdot\frac{(ln)!}{\prod_{r=0}^{n-1}(k-1-rm)!}.$$

As $p>mn$ and $p>ln$, $p$ does not divide $h$ and hence $he\neq 0$. This concludes the proof. □

*Proof of Theorem 1.2.* To avoid triviality, we assume $n\geqslant 2$ and $l=k-1-m(n-1)\geqslant 0$. As $q$ is odd, the norms of those $1-\zeta_s/\zeta_t$ $(1\leqslant s<t\leqslant n)$ (with respect to the field extension $\mathbb{Q}(e^{2\pi i/q})/\mathbb{Q}$) are odd integers and hence $\|\zeta_t^{s-1}\|_{1\leqslant s,t\leqslant n}=\prod_{1\leqslant s<t\leqslant n}(\zeta_t-\zeta_s)$ is not an algebraic integer times two. Therefore $\mathrm{per}(\zeta_t^{s-1})_{1\leqslant s,t\leqslant n}\neq 0$ as observed by Dasgupta et al. [DKSS]. By Lemma 2.1 and (2.6),

$$[x_1^{k-1}\cdots x_n^{k-1}](x_1+\cdots+x_n)^{ln}\prod_{1\leqslant s<t\leqslant n}(\zeta_t x_t-\zeta_s x_s)(x_t-x_s)^{2m-1}\neq 0.$$

Applying Lemma 1.1 we then obtain the desired result. □

*Proof of Theorem 1.3.* For $1\leqslant i<j\leqslant n$, let $r_{ij}$ denote the unique integer in the interval $(-m_{ij}/2,m_{ij}/2]$ which is congruent to $b_i-b_j$ modulo $m_{ij}$. For $x_i\in A_i$ and $x_j\in A_j$, as $|x_i-x_j|<m_{ij}/2$ we have

$$x_i+b_i\equiv x_j+b_j\pmod{m_{ij}}\iff x_j-x_i=r_{ij}.$$

Note also that

$$\mathrm{per}(\alpha_j^{i-1})_{1\leqslant i,j\leqslant n}=\sum_{\sigma\in S_n}\prod_{i=1}^n\alpha_{\sigma(i)}^{i-1}>0.$$

Thus Theorem 1.3 follows from (2.6) with $m = 1$, and Lemmas 1.1 and 2.1. $\square$

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