Combinatorics
Circuit Complexity

Boolean function

\[ f : \{0, 1\}^n \rightarrow \{0, 1\} \]

• DAG (directed acyclic graph)

• Nodes:
  • inputs: \( x_1 \ldots x_n \)
  • gates: \( \land \lor \neg \)

• Complexity: \#gates
**Theorem** (Shannon 1949)

There is a boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ which cannot be computed by any circuit with $\frac{2^n}{3n}$ gates.
# of \( f : \{0, 1\}^n \rightarrow \{0, 1\} \)

\[ \left| \{0, 1\}^{2^n} \right| = 2^{2^n} \]

# of circuits with \( t \) gates:

\[ < 2^t(2n + t + 1)^{2t} \]

- \( \land, \lor \) gates
- \( x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n, 0, 1 \)

De Morgan’s law:
- \( \neg (A \lor B) = \neg A \land \neg B \)
- \( \neg (A \land B) = \neg A \lor \neg B \)

- \( \land, \lor \)
- \( x_i, \neg x_i, 0, 1 \)
- other \((t-1)\) gates
**Theorem** (Shannon 1949)

There is a boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ which cannot be computed by any circuit with $\frac{2^n}{3n}$ gates.

Almost all

one circuit computes one function

#f computable by $t$ gates $\leq$

#circuits with $t$ gates $\leq$

$2^t (2n + t + 1)^{2t} \ll 2^{2^n} = \#f$

$\frac{2^n}{3n} = t$
Double Counting

“Count the same thing twice. The result will be the same.”
Handshaking lemma

A party of \( n \) guests.

The number of guests who shake hands an odd number of times is even.

Modeling:

\[ n \text{ guests} \iff n \text{ vertices} \]

\[ \text{handshaking} \iff \text{edge} \]

\# of handshaking \iff degree
**Lemma (Euler 1736)**

$$\sum_{v \in V} d(v) = 2|E|$$

In the 1736 paper of Seven Bridges of Königsberg

Leonhard Euler
Lemma (Euler 1736)

\[ \sum_{v \in V} d(v) = 2|E| \]

Count directed edges:

\[ (u, v) : \{u, v\} \in E \]

Count by vertex:

\[ \forall v \in V \]
\[ d \text{ directed edges} \]
\[ (v, u_1) \cdots (v, u_d) \]

Count by edge:

\[ \forall \{u, v\} \in E \]
\[ 2 \text{ directions} \]
\[ (u, v) \text{ and } (v, u) \]
**Lemma** (Euler 1736)

\[ \sum_{v \in V} d(v) = 2|E| \]

**Corollary**

# of odd-degree vertices is even.
Sperner’s Lemma

line segment: \( ab \) divided into small segments

each endpoint: red or blue

\( ab \) have different color

\( \exists \) small segment

Emanuel Sperner
Sperner’s Lemma

triangle: $abc$

triangulation

proper coloring:

3 colors red, blue, green

$abc$ is tricolored

lines $ab, bc, ac$ are 2-colored

Sperner’s Lemma (1928)

$\forall$ properly colored triangulation of a triangle,

$\exists$ a tricolored small triangle.
Sperner’s Lemma (1928)

∀ properly colored triangulation of a triangle, ∃ a tricolored small triangle.

partial dual graph:
- each △ is a vertex
- the outer-space is a vertex
- add an edge if 2 △ share a edge

degree of △ node: 1
degree of △ or △ node: 2
other cases: 0 degree
Sperner’s Lemma (1928)

∀ properly colored triangulation of a triangle,
∃ a tricolored small triangle.

**partial dual graph:**

degree of \( \triangle \) node: 1

degree of other \( \triangle \): even

**handshaking lemma:**

\# of odd-degree vertices is even.

\# of \( \triangle \): odd \( \neq 0 \)
Sperner’s Lemma (1928)
∀ properly colored triangulation of a triangle, 
∃ a tricolored small triangle.

Brouwer’s fixed point theorem (1911)
∀ continuous function \( f: B \rightarrow B \) of an \( n \)-dimensional ball \( B \), \( \exists \) a fixed point \( x = f(x) \).

high-dimension: triangle \( \rightarrow \) simplex
triangulation \( \rightarrow \) simplicial subdivision
Pigeonhole Principle

If \( mn \) objects are partitioned into \( n \) classes, then some class receives \( m \) objects.
Schubfachprinzip

“drawer principle”

Dirichlet Principle

Johann Peter Gustav Lejeune Dirichlet
Dirichlet's approximation

$x$ is an irrational number.

Approximate $x$ by a rational with bounded denominator.

Theorem (Dirichlet 1879)

For any natural number $n$, there is a rational number $\frac{p}{q}$ such that $1 \leq q \leq n$ and

\[
\left| x - \frac{p}{q} \right| < \frac{1}{nq}.
\]
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**Theorem (Dirichlet 1879)**

$x$ is an irrational number.

**fractional part:** $\{x\} = x - \lfloor x \rfloor$

**$(n+1)$ pigeons:** $\{kx\}$ for $k = 1, \ldots, n + 1$

**$n$ holes:** $\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right), \ldots, \left(\frac{n-1}{n}, 1\right)$
$x$ is an irrational number.

**fractional part:** \( \{x\} = x - \lfloor x \rfloor \)

**(n+1) pigeons:** \( \{kx\} \) for \( k = 1, \ldots, n+1 \)

**n holes:** \( \left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right), \ldots, \left(\frac{n-1}{n}, 1\right) \)

\[ \exists 1 \leq b < a \leq n+1 \quad \{ax\}, \{bx\} \text{ in the same hole} \]

\[ (a - b)x - ([ax] - [bx]) = \{ax\} - \{bx\} < \frac{1}{n} \]

**integers:** \( q \leq n \quad p \)

\[ |qx - p| < \frac{1}{n} \quad \Rightarrow \quad \left| x - \frac{p}{q} \right| < \frac{1}{nq} \].
An *initiation* question to Mathematics

\[
\forall S \subseteq \{1, 2, \ldots, 2n\} \text{ that } |S| > n \\
\exists a, b \in S \text{ such that } a \mid b
\]

\[
\forall a \in \{1, 2, \ldots, 2n\} \\
a = 2^k m \text{ for an odd } m
\]

\[
C_m = \{2^k m \mid k \geq 0, 2^k m \leq 2n\}
\]

\[
> n \text{ pigeons: } S \\
n \text{ pigeonholes: } C_1, C_3, C_5, \ldots, C_{2n-1}
\]

\[
a < b \quad a, b \in C_m \rightarrow a \mid b
\]
Monotonic subsequences

sequence: \((a_1, \ldots, a_n)\) of \(n\) different numbers

\[1 \leq i_1 < i_2 < \cdots < i_k \leq n\]

subsequence:

\((a_{i_1}, a_{i_2}, \ldots, a_{i_k})\)

increasing:

\[a_{i_1} < a_{i_2} < \ldots < a_{i_k}\]

decreasing:

\[a_{i_1} > a_{i_2} > \ldots > a_{i_k}\]
Theorem (Erdős-Szekeres 1935)

A sequence of \( > mn \) different numbers must contain either an increasing subsequence of length \( m + 1 \), or a decreasing subsequence of length \( n + 1 \).
\((a_1, \ldots, a_N)\) of \(N\) different numbers \(N > mn\)

associate each \(a_i\) with \((x_i, y_i)\)

\(x_i:\) length of longest \textit{increasing} subsequence \textit{ending} at \(a_i\)

\(y_i:\) length of longest \textit{decreasing} subsequence \textit{starting} at \(a_i\)

\(\forall i \neq j, \ (x_i, y_i) \neq (x_j, y_j)\)

\begin{cases}
\text{assume } i < j & \text{Cases.1: } a_i < a_j & \Rightarrow & x_i < x_j \\
\text{Cases.2: } a_i > a_j & \Rightarrow & y_i > y_j
\end{cases}
\((a_1, \ldots, a_N)\) of \(N\) different numbers \(N > mn\)

\(x_i\) : length of longest \textit{increasing} subsequence \textit{ending} at \(a_i\)

\(y_i\) : length of longest \textit{decreasing} subsequence \textit{starting} at \(a_i\)

\(\forall i \neq j, \ (x_i, y_i) \neq (x_j, y_j)\)

“\textit{One pigeon per each hole}.”

No way to put \(N\) pigeons into \(mn\) holes.

\textbf{“\(N\) pigeons”} \((a_1, \ldots, a_N)\)

\(a_i\) is in hole \((x_i, y_i)\)

\(N\)

\(m\)

\(n\)

\(N\)
Theorem (Erdős-Szekeres 1935)

A sequence of \( > mn \) different numbers must contain either an increasing subsequence of length \( m+1 \), or a decreasing subsequence of length \( n+1 \).

\[(a_1, \ldots, a_N) \quad N > mn\]

\(x_i\) : length of longest *increasing* subsequence ending at \(a_i\)

\(y_i\) : length of longest *decreasing* subsequence starting at \(a_i\)