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ON COVERING EQUIVALENCE

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Abstract. An arithmetic sequence $a(n) = \{a+nx : x \in \mathbb{Z}\}$ $(0 \leqslant a < n)$ with weight $\lambda \in \mathbb{C}$ is denoted by $\langle \lambda, a, n\rangle$. For two finite systems $\mathcal{A} = \{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k$ and $\mathcal{B} = \{\langle \mu_t, b_t, m_t\rangle\}_{t=1}^l$ of such triples, if $\sum_{n_s \mid x-a_s} \lambda_s = \sum_{m_t \mid x-b_t} \mu_t$ for all $x \in \mathbb{Z}$ then we say that $\mathcal{A}$ and $\mathcal{B}$ are covering equivalent. In this paper we characterize covering equivalence in various ways, our characterizations involve the Γ-function, the Hurwitz ζ-function, trigonometric functions, the greatest integer function and Egyptian fractions.

1. Introduction and Preliminaries

By $\mathbb{Z}^+$ we mean the set of positive integers. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we let $a+n\mathbb{Z}$ represent the *arithmetic sequence*

$$\{\cdots,\ a-2n,\ a-n,\ a,\ a+n,\ a+2n,\cdots\}$$

and write $a(n)$ for $a+n\mathbb{Z}$ if $a \in R(n) = \{0,1,\cdots,n-1\}$. For a finite system

$$(1.1)\qquad A = \{a_s(n_s)\}_{s=1}^k \qquad (n_s \in \mathbb{Z}^+ \text{ and } a_s \in R(n_s) \text{ for } s = 1,\cdots .k)$$

of such arithmetic sequences, we define the *covering function* $w_A : \mathbb{Z} \to \mathbb{N} = \{0,1,2,\cdots\}$ as follows:

$$(1.2)\qquad w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|.$$

Let m be a positive integer. If $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$, then we call (1.1) an m-cover (of $\mathbb{Z}$); when $w_A(x) = m$ for any $x \in \mathbb{Z}$, we say that A forms an exact m-cover (of $\mathbb{Z}$). The notion of cover (i.e. 1-cover) was introduced by P. Erdös ([Er]) in the early 1930's, one of the simplest nontrivial cover of $\mathbb{Z}$ is $\{0(2),0(3),1(4),5(6),7(12)\}$. Covers of $\mathbb{Z}$ have many nice properties and interesting applications, for problems

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and results we recommend the reader to see R. K. Guy [G], and Š. Porubský and J. Schönheim [PS].

Let M be an additive abelian group. In 1989 the author [Su1] considered triples of the form $\langle\lambda, a, n\rangle$ where $\lambda \in M$, $n \in \mathbb{Z}^+$ and $a \in R(n)$, we can view $\langle\lambda, a, n\rangle$ as the arithmetic sequence $a(n)$ with *weight* (or *multiplier*) λ. Let $\mathrm{S}(M)$ denote the set of all finite systems of such triples. For

$$\mathcal{A} = \{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^k \in \mathrm{S}(M) \tag{1.3}$$

we associate it with the following periodic arithmetical map $w_{\mathcal{A}} : \mathbb{Z} \to M$:

$$w_{\mathcal{A}}(x) = \sum_{\substack{s=1 \\ x\in a_s(n_s)}}^{k} \lambda_s \quad \text{for } x \in \mathbb{Z}, \tag{1.4}$$

which is called the *covering map* of $\mathcal{A}$. ($w_\emptyset$ refers to the zero map into M.) Clearly $w_{\mathcal{A}}$ is periodic modulo the least common multiple $[n_1, \cdots, n_k]$ of all the moduli $n_1, \cdots, n_k$. For $\mathcal{A}, \mathcal{B} \in \mathrm{S}(M)$, if $w_{\mathcal{A}} = w_{\mathcal{B}}$ then we say that $\mathcal{A}$ is *covering equivalent* to $\mathcal{B}$ and write $\mathcal{A} \sim \mathcal{B}$ for this. Observe that (1.3) and $\mathcal{B} = \{\langle\mu_t, b_t, m_t\rangle\}_{t=1}^l \in \mathrm{S}(M)$ are equivalent if and only if

$$C = \{\langle\lambda_1, a_1, n_1\rangle, \cdots, \langle\lambda_k, a_k, n_k\rangle, \langle-\mu_1, b_1, m_1\rangle, \cdots, \langle-\mu_l, b_l, m_l\rangle\} \sim \emptyset. \tag{1.5}$$

We identify (1.1) with the system $\{\langle 1, a_s, n_s\rangle\}_{s=1}^k$. Thus, (1.1) forms an exact m-cover if and only if $A \sim \{\langle m, 0, 1\rangle\}$.

Let M be an additive abelian group, and f a map of two complex variables into M such that $\{\langle\frac{x+r}{n}, ny\rangle : r \in R(n)\} \subseteq \mathrm{Dom}(f)$ for all $\langle x, y\rangle \in \mathrm{Dom}(f)$ and $n \in \mathbb{Z}^+$. If

$$\sum_{r=0}^{n-1} f\left(\frac{x+r}{n}, ny\right) = f(x, y) \quad \text{for any } \langle x, y\rangle \in \mathrm{Dom}(f) \text{ and } n \in \mathbb{Z}^+, \tag{1.6}$$

then we call f a *uniform map* into M. Note that $f(x, y) = f(\frac{x+0}{1}, 1y)$ and $\{r(n)\}_{r=0}^{n-1} \sim \{0(1)\}$ for any $n \in \mathbb{Z}^+$.

In 1989 the author [Su1] showed the following

Theorem 1.1. *Let M be a left R-module where R is a ring with identity. Whenever two systems $\mathcal{A} = \{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^k$ and $\mathcal{B} = \{\langle\mu_t, b_t, m_t\rangle\}_{t=1}^l$ in $\mathrm{S}(R)$ are equivalent, for any uniform map f into M we have*

$$\sum_{s=1}^{k} \lambda_s f\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t f\left(\frac{x+b_t}{m_t}, m_t y\right) \quad \textit{for all } \langle x, y\rangle \in \mathrm{Dom}(f). \tag{1.7}$$

A simple proof of this remarkable theorem was given in Section 2 of [Su8].

Let f be a uniform map into the complex field $\mathbb{C}$ with $\mathrm{Dom}(f) = D \times D'$ and $f(x_0, y_0) \neq 0$ for some $\langle x_0, y_0\rangle \in D \times D'$. If $f(x, y) = g(x)h(y)$ for all $x \in D$ and $y \in D'$, then for each $n \in \mathbb{Z}^+$ we have

$$\sum_{r=0}^{n-1} g\left(\frac{x_0+r}{n}\right) h(ny_0) = g(x_0)h(y_0) \neq 0,$$

thus $\theta(n) = h(y_0)/h(ny_0) \neq 0$ and

$$\theta(n)h(ny) = \frac{h(ny)}{g(x_0)} \sum_{r=0}^{n-1} g\left(\frac{x_0+r}{n}\right) = h(y) \quad \text{for every } y \in D',$$

therefore

$$\theta(mn) = \frac{h(y_0)}{h(my_0)} \cdot \frac{h(my_0)}{h(n(my_0))} = \theta(m)\theta(n) \quad \text{for any } m, n \in \mathbb{Z}^+$$

and

$$\sum_{r=0}^{n-1} g\left(\frac{x+r}{n}\right) = \theta(n)g(x) \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } x \in D. \tag{1.8}$$

Such functions g with $D = \mathrm{Dom}(g) = [0, 1)$ or $\mathbb{Q} \cap [0, 1)$ were studied by H. Walum [W] in 1991 (where $\mathbb{Q}$ denotes the rational field), and investigated earlier by H. Bass [B], D. S. Kubert [K], S. Lang [La] and J. Milnor [M] in the case $\theta(n) = n^{1-s}$ where $s \in \mathbb{C}$.

In this paper by $\mathbb{R}$ and $\mathbb{R}^+$ we mean the field of real numbers and the set of positive reals. For a field F we use F^* to denote the multiplicative group of nonzero elements of F. When $x \in \mathbb{R}$, $[x]$ and $\{x\}$ denote the integral part and the fractional part of x respectively, if $n \in \mathbb{Z}^+$ then we write $\{x\}_n$ to mean $n\{x/n\}$. For convenience, we also set

$$q(0) = 0 \quad \text{and} \quad q(z) = \frac{1}{z} \ \text{ for } z \in \mathbb{C}^*. \tag{1.9}$$

2. Some Examples of Uniform Maps

For a uniform map f, the map $f^-(x, y) = f(1 - x, y)$ is also a uniform map.

Example 2.1. Two simple uniform maps into the additive group $\mathbb{Z}$ are the functions $I, I_+ : \mathbb{C} \times \mathbb{C} \to \mathbb{Z}$ given below:

$$I(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0 & \text{otherwise;} \end{cases} \qquad I_+(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Another example is the function $[\,]:\mathbb{R}\times\mathbb{R}\to\mathbb{Z}$ defined by

$$[\,](x,y)=[x], \tag{2.2}$$

the identity $\sum_{r=0}^{n-1}[\frac{x+r}{n}]=[x]$ (for $n\in\mathbb{Z}^+$) is well known and due to Hermite.

Now we turn to uniform maps into the multiplicative group $\mathbb{C}^*$.

Example 2.2. (i) Define $\gamma:\mathbb{C}\times\mathbb{R}^+\to\mathbb{C}^*$ as follows:

$$\gamma(x,y)=\begin{cases}\Gamma(x)y^x/\sqrt{2\pi y} & \text{if } x\notin -\mathbb{N}=\{0,-1,-2,\cdots\},\\ \frac{(-1)^x}{(-x)!}y^x\sqrt{2\pi y} & \text{otherwise.}\end{cases} \tag{2.3}$$

When $n\in\mathbb{Z}^+$ and $y\in\mathbb{R}^+$, if $x\in\mathbb{C}\setminus -\mathbb{N}$, then by applying Gauss' multiplication formula

$$\prod_{r=0}^{n-1}\Gamma\left(z+\frac{r}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz)$$

with $z=x/n$, we obtain that

$$\prod_{r=0}^{n-1}\left(\Gamma\left(\frac{x+r}{n}\right)\frac{(ny)^{(x+r)/n}}{\sqrt{2\pi ny}}\right)=\Gamma(x)\frac{y^x}{\sqrt{2\pi y}};$$

on the other hand, for $m=kn+l$ with $k\in\mathbb{N}$ and $l\in R(n)$, we have

$$\begin{aligned}\lim_{x\to -m}\prod_{\substack{r=0\\ r\neq l}}^{n-1}\frac{\Gamma(\frac{x+r}{n})(ny)^{\frac{x+r}{n}}}{\sqrt{2\pi ny}}&=\lim_{x\to -m}\frac{\Gamma(x)y^x/\sqrt{2\pi y}}{\Gamma(\frac{x+l}{n})(ny)^{\frac{x+l}{n}}/\sqrt{2\pi ny}}\\ =\lim_{x\to -m}\frac{\Gamma(x+m+1)\prod_{j=0}^{m}(x+j)^{-1}y^{x-1/2}}{\Gamma(\frac{x+m}{n}+1)\prod_{j=0}^{k}(\frac{x+l}{n}+j)^{-1}(ny)^{\frac{x+l}{n}-\frac{1}{2}}}&=\frac{\frac{(-1)^m}{m!}y^{-m}\sqrt{2\pi y}}{\frac{(-1)^k}{k!}(ny)^{-k}\sqrt{2\pi ny}}.\end{aligned}$$

Thus γ is a uniform map into $\mathbb{C}^*$.

(ii) As $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ for $z\notin\mathbb{Z}$, $\gamma(x,y)\gamma^-(x,y)=(-1)^{I_+(x,y)}/S(x,y)$ for $x\in\mathbb{C}$ and $y>0$, where

$$S(x,y)=\begin{cases}2\sin\pi x & \text{if } x\notin\mathbb{Z},\\ (-1)^x y^{-1} & \text{if } x\in\mathbb{Z}.\end{cases} \tag{2.4}$$

It follows that the function $S:\mathbb{C}\times\mathbb{C}^*\to\mathbb{C}^*$ is a uniform map into $\mathbb{C}^*$.

(iii) For $\alpha,\beta,\gamma\in\mathbb{C}$ with $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha+\beta)$ and $\gamma\neq 0,-1,-2,\cdots$, we use $F(\alpha,\beta,\gamma,z)$ to denote the hypergeometric series given by

$$F(\alpha,\beta,\gamma,z)=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}z^n$$

which converges absolutely for $|z| \leqslant 1$. Let $u, v, w \in \mathbb{C}$ and $D_{u,v,w}$ consist of all those $\langle x, y\rangle \in \mathbb{C}\times\mathbb{C}^*$ such that $x+\frac{w}{y}, x+\frac{w-u}{y}, x+\frac{w-v}{y} \notin -\mathbb{N}$ and $\mathrm{Re}(x+\frac{w-u-v}{y}) > 0$. When $\langle x, y\rangle \in D_{u,v,w}$, by a formula in [Ba] we have

$$F\left(\frac{u}{y}, \frac{v}{y}, \frac{w}{y}+x, 1\right) = \frac{\Gamma(x_1)\Gamma(x_1-u/y-v/y)}{\Gamma(x_1-u/y)\Gamma(x_1-v/y)} = \frac{\gamma(x_1,y)\gamma(x_2,y)}{\gamma(x_3,y)\gamma(x_4,y)} \neq 0$$

where $x_1 = x + \frac{w}{y}$, $x_2 = x + \frac{w-u-v}{y}$, $x_3 = x + \frac{w-u}{y}$ and $x_4 = x + \frac{w-v}{y}$; for any $n \in \mathbb{Z}^+$ clearly $\langle \frac{x+r}{n}, ny\rangle \in D_{u,v,w}$ for all $r \in R(n)$ (since $\frac{x+r}{n} + \frac{z}{ny} = \frac{x+z/y+r}{n}$) and

$$\begin{aligned}&\prod_{r=0}^{n-1} F\left(\frac{u}{ny}, \frac{v}{ny}, \frac{w}{ny}+\frac{x+r}{n}, 1\right) = \prod_{r=0}^{n-1} F\left(\frac{u}{ny}, \frac{v}{ny}, \frac{x_1+r}{n}, 1\right)\\ =&\prod_{r=0}^{n-1} \frac{\gamma(\frac{x_1+r}{n}, ny)\gamma(\frac{x_2+r}{n}, ny)}{\gamma(\frac{x_3+r}{n}, ny)\gamma(\frac{x_4+r}{n}, ny)} = \frac{\gamma(x_1,y)\gamma(x_2,y)}{\gamma(x_3,y)\gamma(x_4,y)} = F\left(\frac{u}{y}, \frac{v}{y}, \frac{w}{y}+x, 1\right).\end{aligned}$$

So the function $F_{u,v,w} : D_{u,v,w} \to \mathbb{C}^*$ given by

$$F_{u,v,w}(x,y) = F\left(\frac{u}{y}, \frac{v}{y}, \frac{w}{y}+x, 1\right) \tag{2.5}$$

is a uniform map into $\mathbb{C}^*$.

Remark 2.1. Since the function S is a uniform map into $\mathbb{C}^*$, we have

$$\sum_{r=0}^{n-1} \log\left(2\sin\pi\frac{x+r}{n}\right) = \log(2\sin\pi x) \quad \text{for } n \in \mathbb{Z}^+ \text{ and } x \in (0,1).$$

In 1966 H. Bass [B] showed that every linear relation over $\mathbb{Q}$ among the numbers $\log(2\sin\pi x)$ with $x \in \mathbb{Q}\cap(0,1)$, is a consequence of the last identity, together with the fact that $\log(2\sin\pi(1-x)) = \log(2\sin\pi x)$. (See also V. Ennola [E].)

Example 2.3. (i) Define $G, H : \mathbb{C}\times\mathbb{R}^+ \to \mathbb{C}$ by

$$G(x,y) = \frac{1}{y}\left(\log y + \sum_{m=0}^{\infty}\left(\frac{1}{m+1} - q(m+x)\right)\right) \tag{2.6}$$

and

$$H(x,y) = \frac{1}{y}\left(\log y + \sum_{m=1}^{\infty}\left(\frac{1}{m} + q(x-m)\right)\right). \tag{2.7}$$

We assert that G is a uniform map into $\mathbb{C}$ and hence so is $H = G^-$.

Let $n\in\mathbb{Z}^+$ and $y>0$. By Example 2.2(i),

$$\prod_{r=0}^{n-1}\left(\Gamma\left(\frac{x+r}{n}\right)(ny)^{\frac{x+r}{n}}(2\pi ny)^{-\frac12}\right)=\Gamma(x)y^x(2\pi y)^{-\frac12}\quad\text{for } x\in\mathbb{C}\setminus-\mathbb{N}.$$

Taking the logarithmic derivatives of both sides we get that

$$\frac1n\sum_{r=0}^{n-1}\left(\frac{\Gamma'((x+r)/n)}{\Gamma((x+r)/n)}+\log(ny)\right)=\frac{\Gamma'(x)}{\Gamma(x)}+\log y\quad\text{for } x\neq 0,-1,-2,\cdots.$$

Thus

$$\frac{1}{ny}\sum_{r=0}^{n-1}\left(\frac{\Gamma'((x+r)/n)}{\Gamma((x+r)/n)}+\log(ny)+\gamma\right)=\frac1y\left(\frac{\Gamma'(x)}{\Gamma(x)}+\log y+\gamma\right)\ \text{for } x\in\mathbb{C}\setminus-\mathbb{N}$$

where γ is the Euler constant. By a known formula (see, e.g., [Ba]), if $x\in\mathbb{C}\setminus-\mathbb{N}$ then

$$yG(x,y)-\log y=\sum_{m=0}^{\infty}\left(\frac{1}{m+1}-\frac{1}{m+x}\right)=\sum_{m=0}^{\infty}\frac{x-1}{(m+1)(m+x)}=\frac{\Gamma'(x)}{\Gamma(x)}+\gamma.$$

So, for any $x\in\mathbb{C}\setminus-\mathbb{N}$ we have

$$\sum_{r=0}^{n-1}G\left(\frac{x+r}{n},ny\right)=G(x,y).$$

When $x=-a-bn$ where $a\in R(n)$ and $b\in\mathbb{N}$,

$$\begin{aligned}
&\sum_{r=0}^{n-1}G\left(\frac{x+r}{n},ny\right)-G(x,y)\\
=&G\left(\frac{x+a}{n},ny\right)-G(x,y)+\lim_{\substack{z\to x\\ z\notin\mathbb{Z}}}\sum_{\substack{r=0\\ r\neq a}}^{n-1}G\left(\frac{z+r}{n},ny\right)\\
=&G(-b,ny)-G(x,y)+\lim_{\substack{z\to x\\ z\notin\mathbb{Z}}}\left(G(z,y)-G\left(\frac{z+a}{n},ny\right)\right)\\
=&\lim_{\substack{z\to x\\ z\notin\mathbb{Z}}}\left(\frac1y\sum_{\substack{m=0\\ m\neq -x}}^{\infty}(q(m+x)-q(m+z))+\frac1y(q(0)-q(-x+z))\right.\\
&\qquad\left.+\frac{1}{ny}\sum_{\substack{m=0\\ m\neq b}}^{\infty}\left(q\left(m+\frac{z+a}{n}\right)-q(m-b)\right)+\frac{1}{ny}\left(q\left(b+\frac{z+a}{n}\right)-q(0)\right)\right)\\
=&\lim_{z\to x}\left(\frac1y\cdot\frac{1}{x-z}+\frac{1}{ny}\cdot\frac{1}{(z+a)/n+b}\right)=0.
\end{aligned}$$

(Note that for any $v \in \mathbb{N}$ we have

$$\lim_{u\to v}\sum_{\substack{m=0\\ m\neq v}}^{\infty}\left(\frac{1}{m-u}-\frac{1}{m-v}\right)=\sum_{m=v+1}^{\infty}\lim_{u\to v}\frac{u-v}{(m-u)(m-v)}=0.$$

For, if $|u-v|<1/2$ then

$$\left|\frac{1}{m-u}-\frac{1}{m-v}\right|=\left|\frac{u-v}{(m-u)(m-v)}\right|<\frac{1/2}{(m-v-1/2)^2}$$

for $m=v+1, v+2, \cdots$, thus the series $\sum_{m=v+1}^{\infty}(\frac{1}{m-u}-\frac{1}{m-v})$ (with $|u-v|<1/2$) converges uniformly.)

(ii) The Hurwitz zeta function $\zeta(s,v)$ is defined by the series

$$\zeta(s,v)=\sum_{m=0}^{\infty}\frac{1}{(m+v)^s}$$

for $s, v \in \mathbb{C}$ with $\mathrm{Re}(s)>1$ and $\mathrm{Re}(v)>0$ (or $v\notin -\mathbb{N}$), by Lemma 1 of [M] it extends to a function which is defined and holomorphic in both variables for all complex $s\neq 1$ and for all v in the simply connected region $\mathbb{C}\setminus(-\infty,0]$. In addition, we set $\zeta(1,v)=-G(v,1)$. For $v\in\mathbb{C}\setminus -\mathbb{N}$, if $\mathrm{Re}(s)>1$ then

$$\zeta(s,v+1)-\zeta(s,v)=\sum_{m=0}^{\infty}\frac{1}{(m+v+1)^s}-\sum_{m=0}^{\infty}\frac{1}{(m+v)^s}=-\frac{1}{v^s};$$

we also have

$$\zeta(1,v+1)-\zeta(1,v)=-\sum_{m=1}^{\infty}\left(\frac{1}{m}-\frac{1}{m+v}\right)+\sum_{m=0}^{\infty}\left(\frac{1}{m+1}-\frac{1}{m+v}\right)=-\frac{1}{v}.$$

If $\mathrm{Re}(s)>1$ then

$$\frac{d}{dv}\zeta(s,v)=\sum_{m=0}^{\infty}\frac{d(m+v)^{-s}}{dv}=-s\zeta(s+1,v)\quad\text{for all } v\in\mathbb{C}\setminus-\mathbb{N}.$$

Whenever $s\in\mathbb{C}\setminus\{0,1\}$,

$$\frac{d}{dv}\zeta(s,v)=-s\zeta(s+1,v)\quad\text{for all } v\in\mathbb{C}\setminus(-\infty,0]$$

by analytic continuation. As $\zeta(0,v)=\frac{1}{2}-v$, $\frac{d}{dv}\zeta(0,v)=-1$. For those $v\in\mathbb{C}\setminus-\mathbb{N}$, we have

$$\frac{d}{dv}\zeta(1,v)=-\frac{d}{dv}\sum_{m=0}^{\infty}\left(\frac{1}{m+1}-\frac{1}{m+v}\right)=-\sum_{m=0}^{\infty}\frac{1}{(m+v)^2}=-\zeta(2,v).$$

For $s \in \mathbb{C}$ we define $\zeta_s : (\mathbb{C} \setminus (-\infty, 0]) \times \mathbb{R}^+ \to \mathbb{C}$ by

$$\zeta_s(x,y) = \begin{cases} y^{-s}\zeta(s,x) & \text{if } s \neq 1, \\ y^{-1}(\zeta(1,x) - \log y) & \text{if } s = 1. \end{cases} \tag{2.8}$$

Let $x \in \mathbb{C} \setminus (-\infty, 0]$ and $y \in \mathbb{R}^+$. Then $\zeta_1(x,y) = -G(x,y)$. If $\mathrm{Re}(s) > 1$ and $n \in \mathbb{Z}^+$, then

$$\sum_{r=0}^{n-1} n^{-s}\zeta\left(s, \frac{x+r}{n}\right) = \sum_{r=0}^{n-1}\sum_{k=0}^{\infty} \frac{1}{(x+r+kn)^s} = \zeta(s,x).$$

By part (i) and analytic continuation, for any $s \in \mathbb{C}$ the function ζ_s is a uniform map into $\mathbb{C}$.

Remark 2.2. (a) In [M] Milnor observed that there is no constant c such that

$$\sum_{r=0}^{n-1} \frac{1}{n}\left(\frac{\Gamma'((x+r)/n)}{\Gamma((x+r)/n)} + c\right) = \frac{\Gamma'(x)}{\Gamma(x)} + c \quad \text{for } n \in \mathbb{Z}^+ \text{ and } x \in (0,1).$$

(b) By [Ba], if $x, y > 0$ then

$$\begin{aligned} \frac{d\zeta_s(x,y)}{ds}\bigg|_{s=0} &= y^{-s}\frac{d}{ds}\zeta(s,x)\bigg|_{s=0} - y^{-s}\log y \cdot \zeta(s,x)\bigg|_{s=0} \\ &= \log\Gamma(x) - \frac{1}{2}\log(2\pi) + \left(x - \frac{1}{2}\right)\log y = \log\gamma(x,y). \end{aligned}$$

(c) For $s \in \mathbb{C} \setminus -\mathbb{N}$ Milnor [M] proved in 1983 that the complex vector space of all those continuous functions $g : (0,1) \to \mathbb{C}$ which satisfy (1.8) with $D = (0,1)$ and $\theta(n) = n^s$, is spanned by linearly independent functions $\zeta(s,x)$ and $\zeta(s, 1-x)$ where x ranges over $(0,1)$.

Example 2.4. (i) Let ψ be a function into $\mathbb{C}$ with $\mathrm{Dom}(\psi) \subseteq \mathbb{C}$. Let D_ψ denote the set of those $\langle x, y\rangle \in \mathbb{C} \times \mathbb{C}$ such that $(x+k)y \in \mathrm{Dom}(\psi)$ for all $k \in \mathbb{Z}$ and $\sum_{k \equiv a \pmod n} \psi((x+k)y))$ converges for any $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. If $\langle x, y\rangle \in D_\psi$ and $n \in \mathbb{Z}^+$, then $\langle \frac{x+r}{n}, ny\rangle \in D_\psi$ for all $r \in R(n)$ since $(\frac{x+r}{n} + k)ny = (x + (r+kn))y$, furthermore

$$\sum_{r=0}^{n-1}\sum_{k=-\infty}^{\infty} \psi\left(\left(\frac{x+r}{n} + k\right)ny\right) = \sum_{r=0}^{n-1}\sum_{l \equiv r \pmod n} \psi((x+l)y) = \sum_{l=-\infty}^{+\infty} \psi((x+l)y)$$

Thus the function $\tilde{\psi} : D_\psi \to \mathbb{C}$ given by

$$\tilde{\psi}(x,y) = \sum_{k=-\infty}^{+\infty} \psi(xy + ky) \tag{2.9}$$

is a uniform map into $\mathbb{C}$. This fact was first mentioned by the author in [S1].

(ii) Let $m \in \mathbb{N}$. We define $\cot_m : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$ by

$$\cot_m(x, y) = \begin{cases} \frac{(-1)^m}{y^{m+1}} \cot^{(m)}(\pi x) & \text{if } x \notin \mathbb{Z}, \\ (-1)^{\frac{m-1}{2}} (\frac{2}{y})^{m+1} \frac{B_{m+1}}{m+1} & \text{if } x \in \mathbb{Z} \ \&\ 2 \nmid m, \\ 0 & \text{if } x \in \mathbb{Z} \ \&\ 2 \mid m, \end{cases} \tag{2.10}$$

where $\cot^{(m)}(z) = \frac{d^m \cot z}{dz^m}$ for $z \in \mathbb{C} \setminus \pi\mathbb{Z}$, and B_n is the nth Bernoulli number. Fix $x \in \mathbb{C}$ and $y \in \mathbb{C}^*$. As

$$\pi \cot \pi v = \sum_{k=-\infty}^{+\infty} \frac{1}{v+k} = \frac{1}{v} + 2v \sum_{k=1}^{\infty} \frac{1}{v^2 - k^2} \quad \text{for } v \in \mathbb{C} \setminus \mathbb{Z},$$

if $x \notin \mathbb{Z}$ then

$$\pi^{m+1} \cot^{(m)}(\pi x) = \frac{d^m}{dv^m}\left(\frac{1}{v}\right)\Big|_{v=x} + \sum_{k=1}^{\infty} \frac{d^m}{dv^m}\left(\frac{2v}{v^2-k^2}\right)\Big|_{v=x}$$
$$= \sum_{k=-\infty}^{+\infty} \frac{d^m}{dv^m}\left(\frac{1}{v+k}\right)\Big|_{v=x} = (-1)^m m! \sum_{k=-\infty}^{+\infty} \frac{1}{(x+k)^{m+1}}$$

and so

$$\cot_m(x, y) = \frac{m!}{(\pi y)^{m+1}} \sum_{k=-\infty}^{+\infty} \frac{1}{(x+k)^{m+1}}.$$

If $x \in \mathbb{Z}$ and $2 \mid m$, then $\cot_m(x, y)$ vanishes and

$$\sum_{\substack{k=-\infty \\ k \neq -x}}^{+\infty} \frac{1}{(x+k)^{m+1}} = \sum_{\substack{l=-\infty \\ l \neq 0}}^{+\infty} \frac{1}{l^{m+1}} = 0.$$

If $x \in \mathbb{Z}$ and $2 \nmid m$, then

$$\cot_m(x, y) = (-1)^{\frac{m+1}{2}-1} \frac{(2\pi)^{m+1}}{(m+1)!} B_{m+1} \frac{m!}{(\pi y)^{m+1}} = \frac{m! 2\zeta(m+1)}{(\pi y)^{m+1}}$$
$$= \frac{m!}{(\pi y)^{m+1}} \sum_{\substack{l=-\infty \\ l \neq 0}}^{+\infty} \frac{1}{l^{m+1}} = \frac{m!}{(\pi y)^{m+1}} \sum_{\substack{k=-\infty \\ k \neq -x}}^{+\infty} \frac{1}{(x+k)^{m+1}}.$$

So we always have

$$\cot_m(x, y) = \frac{m!}{(\pi y)^{m+1}} \sum_{\substack{k=-\infty \\ k \neq -x}}^{+\infty} \frac{1}{(x+k)^{m+1}} = \frac{m!}{\pi^{m+1}} \sum_{k=-\infty}^{+\infty} q((xy+ky)^{m+1}).$$

By part (i) this implies that $\cot_m$ is a uniform map into $\mathbb{C}$.

Let $x \in \mathbb{C}$ and $y > 0$. Clearly

$$\begin{aligned}\pi\cot_0(x,y) =& \frac{1-I(x,y)}{y}\pi\cot\pi x \\ =& H(x,y) - H^-(x,y) = \zeta_1(x,y) - \zeta_1^-(x,y).\end{aligned} \tag{2.11}$$

Also,

$$\cot_1(x,y) = \begin{cases} \frac{1}{y^2}\csc^2\pi x & \text{if } x \notin \mathbb{Z}, \\ \frac{1}{3y^2} & \text{if } x \in \mathbb{Z}; \end{cases} \tag{2.12}$$

$$\cot_2(x,y) = \begin{cases} \frac{2\cos\pi x}{y^3\sin^3\pi x} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases} = 2\cot_0(x,y)\cot_1(x,y); \tag{2.13}$$

and

$$\cot_3(x,y) = \begin{cases} \frac{2(1+2\cos^2\pi x)}{y^4\sin^4\pi x} & \text{if } x \notin \mathbb{Z}, \\ \frac{2}{15y^4} & \text{if } x \in \mathbb{Z}, \end{cases} = \frac{6}{(\pi y)^4}\sum_{\substack{k=-\infty \\ k \neq -x}}^{+\infty} \frac{1}{(x+k)^4}. \tag{2.14}$$

Remark 2.3. In 1970 S. Chowla [C] proved that if p is an odd prime, then the $\frac{p-1}{2}$ real numbers $\cot 2\pi\frac{r}{p}$ $(r = 1, 2, \cdots, \frac{p-1}{2})$ are linearly independent over $\mathbb{Q}$, this was extended by T. Okada [O] in 1980. By Lemma 7 of Milnor [M], there is a unique function $g : \mathbb{R} \to \mathbb{R}$ periodic mod 1 for which $g(x) = \cot^{(m)}(\pi x) = (-1)^m\cot_m(x,1)$ for $x \in \mathbb{R} \setminus \mathbb{Z}$ and (1.8) holds with $\theta(n) = n^{m+1}$ and $D = \mathbb{R}$, we remark that $g(x) = (-1)^m\cot_m(x,1)$ for all $x \in \mathbb{R}$ because $\cot_m$ is a uniform map into $\mathbb{C}$. With the help of Dirichlet L-functions, Milnor [M] also showed that every $\mathbb{Q}$-linear relation among the values $g(x)$ with $x \in \mathbb{Q}$, follows from (1.8) with $\theta(n) = n^{m+1}$ and $D = \mathbb{Q}$, and the facts $g(x+1) = g(x)$ and $g(-x) = (-1)^{m+1}g(x)$ for $x \in \mathbb{Q}$. So, each $\mathbb{Q}$-linear relation among the values $\cot_m(x,n) = \frac{(-1)^m}{n^{m+1}}g(x)$ with $x \in \mathbb{Q}$ and $n \in \mathbb{Z}^+$, is a consequence of the fact that $\cot_m$ is a uniform map into $\mathbb{C}$, together with the trivial equalities $\cot_m(x+1,n) = \cot_m(x,n)$ and $\cot_m^- = (-1)^{m-1}\cot_m$.

3. Characterizations of covering equivalence

In order to characterize covering equivalence, we need

Lemma 3.1. *Let f be a complex-valued function so that for any $n \in \mathbb{Z}^+$, $f(x,n)$ is defined and continuous at $x \in (-\infty, 1) \setminus \mathbb{Z}$, and*

$$\sum_{r=0}^{n-1} f\left(\frac{x+r}{n}, n\right) = f(x,1) \quad \text{for all } x < 1 \text{ with } x \notin \mathbb{Z}.$$

Suppose that

$$\lim_{x\to -m} f(x,1)=\infty \qquad \textit{for each } m\in\mathbb{N}.$$

Let $\mathcal{A}=\{\langle\lambda_s,a_s,n_s\rangle\}_{s=1}^k$ *be such a system in* $\mathrm{S}(\mathbb{C})$ *that*

$$\sum_{s=1}^k \lambda_s f\left(\frac{x+a_s}{n_s},n_s\right)=0 \qquad \textit{for all } x<1 \textit{ with } x\notin\mathbb{Z}.$$

Then $\mathcal{A}\sim\emptyset$.

Proof. Since $w_{\mathcal{A}}$ is periodic mod $[n_1,\cdots,n_k]$, it suffices to show $w_{\mathcal{A}}(m)=0$ for any $m\in\mathbb{N}$. As $\lim_{x\to -m} f(x,1)=\infty$ there exists a $\delta\in(0,1)$ such that $f(x,1)\neq 0$ for all $x\in(-m-\delta,-m+\delta)$ with $x\neq -m$. If $n\in\mathbb{Z}^+$ then

$$\begin{aligned}&\lim_{x\to -m}\left(f(x,1)-f\left(\frac{x+\{m\}_n}{n},n\right)\right)\\ =&\lim_{x\to -m}\sum_{\substack{r=0\\ r\neq\{m\}_n}}^{n-1} f\left(\frac{x+r}{n},n\right)=\sum_{\substack{r=0\\ r\neq\{m\}_n}}^{n-1} f\left(\frac{-m+r}{n},n\right)\end{aligned}$$

and hence

$$f\left(\frac{x+\{m\}_n}{n},n\right)\sim f(x,1) \qquad \text{as } x\to -m.$$

So

$$\lim_{x\to -m}\frac{f(\frac{x+a_s}{n_s},n_s)}{f(x,1)}=\begin{cases}1 & \text{if } n_s\mid m-a_s,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore

$$w_{\mathcal{A}}(m)=\sum_{s=1}^k\lambda_s\lim_{x\to -m}\frac{f(\frac{x+a_s}{n_s},n_s)}{f(x,1)}=\lim_{x\to -m}\frac{1}{f(x,1)}\sum_{s=1}^k\lambda_s f\left(\frac{x+a_s}{n_s},n_s\right)=0.$$

Let's now characterize the covering equivalence of two systems of arithmetic sequences.

Theorem 3.1. *Let* $n_s,m_t\in\mathbb{Z}^+$, $a_s\in R(n_s)$ *and* $b_t\in R(m_t)$ *for* $s=1,\cdots,k$ *and* $t=1,\cdots,l$. *Then the following statements are equivalent:*

$$A=\{a_s(n_s)\}_{s=1}^k\sim B=\{b_t(m_t)\}_{t=1}^l; \tag{3.1}$$

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \sum_{s\in I}\frac{1}{n_s}=c}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}\frac{a_s}{n_s}}=\sum_{\substack{J\subseteq\{1,\cdots,l\}\\ \sum_{t\in J}\frac{1}{m_t}=c}}(-1)^{|J|}e^{2\pi i\sum_{t\in J}\frac{b_t}{m_t}} \quad \textit{for all } c\geqslant 0; \tag{3.2}$$

(3.3)
$$2^k\prod_{\substack{s=1\\ s\notin S_z}}^k \sin\pi\frac{a_s-z}{n_s}\cdot\prod_{s\in S_z}\frac{(-1)^{[\frac{z}{n_s}]}}{n_s}=2^l\prod_{\substack{t=1\\ t\notin T_z}}^l \sin\pi\frac{b_t-z}{m_t}\cdot\prod_{t\in T_z}\frac{(-1)^{[\frac{z}{m_t}]}}{m_t}\ \textit{for } z\in\mathbb{C}$$

where $S_z=\{1\leqslant s\leqslant k:\ z\in a_s(n_s)\}$ *and* $T_z=\{1\leqslant t\leqslant l:\ z\in b_t(m_t)\}$*;*

$$(3.4)\quad \frac{\prod_{\substack{s=1\\ s\notin U_z}}^k\Gamma\left(\frac{a_s-z}{n_s}\right)n_s^{\frac{a_s-z}{n_s}-\frac12}}{\prod_{\substack{t=1\\ t\notin V_z}}^l\Gamma\left(\frac{b_t-z}{m_t}\right)m_t^{\frac{b_t-z}{m_t}-\frac12}}=(2\pi)^{\frac{k-l}{2}}\frac{\prod_{s\in U_z}[\frac{z}{n_s}]!(-1)^{[\frac{z}{n_s}]}n_s^{[\frac{z}{n_s}]-\frac12}}{\prod_{t\in V_z}[\frac{z}{m_t}]!(-1)^{[\frac{z}{m_t}]}m_t^{[\frac{z}{m_t}]-\frac12}}\ \textit{for } z\in\mathbb{C}$$

where $U_z=\{1\leqslant s\leqslant k:\ z\in a_s+n_s\mathbb{N}\}$ *and* $V_z=\{1\leqslant t\leqslant l:\ z\in b_t+m_t\mathbb{N}\}$*;*

$$(3.5)\quad \prod_{s=1}^k F\left(\frac{u}{n_s},\frac{v}{n_s},\frac{w+a_s}{n_s},1\right)=\prod_{t=1}^l F\left(\frac{u}{m_t},\frac{v}{m_t},\frac{w+b_t}{m_t},1\right)$$
$$\textit{for } u,v,w\in\mathbb{C}\ \textit{with } \mathrm{Re}(w)>\mathrm{Re}(u+v)\ \textit{and } w,w-u,w-v\notin-\mathbb{N}.$$

Proof. (3.1) $\Leftrightarrow$ (3.2). Let $N=[n_1,\cdots,n_k,m_1,\cdots,m_l]$. Set
$$f(z)=\prod_{s=1}^k\left(1-z^{N/n_s}e^{2\pi i a_s/n_s}\right)\quad\text{and}\quad g(z)=\prod_{t=1}^l\left(1-z^{N/m_t}e^{2\pi i b_t/m_t}\right).$$

Clearly any zero of $f(z)$ or $g(z)$ is an Nth root of unity. For each $a\in\mathbb{Z}$, $e^{2\pi i a/N}$ is a zero of $f(z)$ with multiplicity $w_A(-a)$, and a zero of $g(z)$ with multiplicity $w_B(-a)$. By Viéte's theorem, we have the identity $f(z)=g(z)$ if and only if $w_A=w_B$. Note that $f(z)=g(z)$ if and only if
$$\sum_{I\subseteq\{1,\cdots,k\}}(-1)^{|I|}z^{\sum_{s\in I}N/n_s}e^{2\pi i\sum_{s\in I}a_s/n_s}$$
$$=\sum_{J\subseteq\{1,\cdots,l\}}(-1)^{|J|}z^{\sum_{t\in J}N/m_t}e^{2\pi i\sum_{t\in J}b_t/m_t}.$$

By comparing the coefficients of powers of z, we find that $w_A=w_B$ if and only if
$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \sum_{s\in I}N/n_s=a}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}a_s/n_s}=\sum_{\substack{J\subseteq\{1,\cdots,k\}\\ \sum_{t\in J}N/m_t=a}}(-1)^{|J|}e^{2\pi i\sum_{t\in J}b_t/m_t}$$

for all $a=0,1,2,\cdots$. This proves the equivalence of (3.1) and (3.2).

(3.1) $\Rightarrow$ (3.3),(3.4). We can view the multiplicative group $\mathbb{C}^*$ as a $\mathbb{Z}$-module with the scalar product $\langle m,z\rangle\mapsto z^m$. By Theorem 1.1, for any $z\in\mathbb{C}$ we have
$$\prod_{s=1}^k S\left(\frac{a_s-z}{n_s},n_s\right)=\prod_{t=1}^l S\left(\frac{b_t-z}{m_t},m_t\right)$$

and

$$\prod_{s=1}^{k}\gamma\left(\frac{a_s-z}{n_s},n_s\right)=\prod_{t=1}^{l}\gamma\left(\frac{b_t-z}{m_t},m_t\right).$$

Apparently $|S_z|=|T_z|$ and $|U_z|=|V_z|$. If $n\in\mathbb{Z}^+$, $a\in R(n)$ and $z\in a(n)$, then $-\frac{a-z}{n}=\frac{z-a}{n}=[\frac{z}{n}]$. Therefore (3.3) and (3.4) follow.

(3.3)⇒(3.1), and (3.4)⇒(3.1). For $n\in\mathbb{Z}^+$ and $x\in(-\infty,1)\setminus\mathbb{Z}$, we put

$$f_1(x,n)=\log|2\sin\pi x|,\quad f_2(x,n)=\log|\Gamma(x)|+\left(x-\frac{1}{2}\right)\log n-\frac{1}{2}\log(2\pi).$$

Let $j\in\{1,2\}$. Then $\lim_{x\to -m}f_j(x,1)=\infty$ for all $m\in\mathbb{N}$. Let $n\in\mathbb{Z}^+$. Then $f_j(x,n)$ is continuous for $x\in(-\infty,1)\setminus\mathbb{Z}$. When $x<1$ and $x\notin\mathbb{Z}$,

$$\sum_{r=0}^{n-1}f_1\left(\frac{x+r}{n},n\right)=\log\left|\prod_{r=0}^{n-1}\left(2\sin\pi\frac{x+r}{n}\right)\right|=\log|2\sin\pi x|=f_1(x,1)$$

and

$$\sum_{r=0}^{n-1}f_2\left(\frac{x+r}{n},n\right)=\log\left|\prod_{r=0}^{n-1}\left(\Gamma\left(\frac{x+r}{n}\right)\frac{n^{\frac{x+r}{n}-\frac{1}{2}}}{\sqrt{2\pi}}\right)\right|=\log\left|\frac{\Gamma(x)}{\sqrt{2\pi}}\right|=f_2(x,1).$$

If

$$\prod_{s=1}^{k}\left(2\sin\pi\frac{x+a_s}{n_s}\right)\cdot\prod_{t=1}^{l}\left(2\sin\pi\frac{x+b_t}{m_t}\right)^{-1}=1$$

for $x\in(-\infty,1)\setminus\mathbb{Z}$, or

$$\prod_{s=1}^{k}\left(\Gamma\left(\frac{x+a_s}{n_s}\right)\frac{n_s^{\frac{x+a_s}{n_s}-\frac{1}{2}}}{\sqrt{2\pi}}\right)\cdot\prod_{t=1}^{l}\left(\Gamma\left(\frac{x+b_t}{m_t}\right)\frac{m_t^{\frac{x+b_t}{m_t}-\frac{1}{2}}}{\sqrt{2\pi}}\right)^{-1}=1$$

for $x\in(-\infty,1)\setminus\mathbb{Z}$, then for $j=1$ or 2 we have

$$\sum_{s=1}^{k}f_j\left(\frac{x+a_s}{n_s},n_s\right)-\sum_{t=1}^{l}f_j\left(\frac{x+b_t}{m_t},m_t\right)=0\ \text{ for all } x<1 \text{ with } x\notin\mathbb{Z},$$

therefore $\{\langle 1,a_1,n_1\rangle,\cdots,\langle 1,a_k,n_k\rangle,\langle -1,b_1,m_1\rangle,\cdots,\langle -1,b_l,m_l\rangle\}\sim\emptyset$ by Lemma 3.1. So, each of (3.3) and (3.4) implies (3.1).

(3.1) ⇒ (3.5). Let u,v,w be complex numbers with $\mathrm{Re}(w)>\mathrm{Re}(u+v)$ and $w,w-u,w-v\notin-\mathbb{N}$. By Example 2.2(iii), $F_{u,v,w}$ is a uniform map into the multiplicative group $\mathbb{C}^*$. Note that $\langle 0,1\rangle\in D_{u,v,w}$. Since $A\sim B$, applying Theorem 1.1 we get that

$$\prod_{s=1}^{k}F_{u,v,w}\left(\frac{a_s}{n_s},n_s\right)=\prod_{t=1}^{l}F_{u.v.w}\left(\frac{b_t}{m_t},m_t\right),$$

i.e.,

$$\prod_{s=1}^{k} F\left(\frac{u}{n_s}, \frac{v}{n_s}, \frac{w+a_s}{n_s}, 1\right) = \prod_{t=1}^{l} F\left(\frac{u}{m_t}, \frac{v}{m_t}, \frac{w+b_t}{m_t}, 1\right).$$

(3.5) $\Rightarrow$ (3.1). Let $x \in \mathbb{R} \setminus \mathbb{Z}$. By a known formula in [Ba], if $n \in \mathbb{Z}^+$ and $a \in R(n)$ then

$$\begin{aligned} F\left(\frac{\frac{1}{2}-x}{n}, \frac{x-\frac{1}{2}}{n}, \frac{\frac{1}{2}+a}{n}, 1\right) &= \frac{\Gamma\left(\frac{\frac{1}{2}+a}{n}\right)\Gamma\left(\frac{\frac{1}{2}+a}{n} - \frac{\frac{1}{2}-x}{n} - \frac{x-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{\frac{1}{2}+a}{n} - \frac{\frac{1}{2}-x}{n}\right)\Gamma\left(\frac{\frac{1}{2}+a}{n} - \frac{x-\frac{1}{2}}{n}\right)} \\ &= \frac{\Gamma^2\left(\frac{1+2a}{2n}\right)}{\Gamma\left(\frac{x+a}{n}\right)\Gamma\left(\frac{1-x+a}{n}\right)}. \end{aligned}$$

So, by (3.5) we have

$$\prod_{s=1}^{k} \frac{\Gamma^2\left(\frac{1+2a_s}{2n_s}\right)}{\Gamma\left(\frac{x+a_s}{n_s}\right)\Gamma\left(\frac{1-x+a_s}{n_s}\right)} = \prod_{t=1}^{l} \frac{\Gamma^2\left(\frac{1+2b_t}{2m_t}\right)}{\Gamma\left(\frac{x+b_t}{m_t}\right)\Gamma\left(\frac{1-x+b_t}{m_t}\right)},$$

i.e.,

$$(*) \qquad \frac{\prod_{s=1}^{k}\Gamma\left(\frac{x+a_s}{n_s}\right)}{\prod_{t=1}^{l}\Gamma\left(\frac{x+b_t}{n_s}\right)} = \frac{\prod_{s=1}^{k}\left(\Gamma^2\left(\frac{1+2a_s}{2n_s}\right)/\Gamma\left(\frac{1-x+a_s}{n_s}\right)\right)}{\prod_{t=1}^{l}\left(\Gamma^2\left(\frac{1+2b_t}{2m_t}\right)/\Gamma\left(\frac{1-x+b_t}{m_t}\right)\right)}.$$

Note that $\Gamma(1+z) = z\Gamma(z) = z(z-1)\cdots(z-n)\Gamma(z-n)$ for $n \in \mathbb{N}$ and $z \notin \mathbb{Z}$. Let $m \in \mathbb{N}$. Then

$$(\star) \qquad \frac{\prod_{\substack{1\leqslant s\leqslant k \\ n_s\mid m-a_s}}\left(\Gamma\left(1+\frac{x+m}{n_s}\right)/\prod_{j=0}^{[m/n_s]}\left(\frac{x+m}{n_s}-j\right)\right)}{\prod_{\substack{1\leqslant t\leqslant l \\ m_t\mid m-b_t}}\left(\Gamma\left(1+\frac{x+m}{m_t}\right)/\prod_{j=0}^{[m/m_t]}\left(\frac{x+m}{m_t}-j\right)\right)} = \frac{\prod_{\substack{1\leqslant s\leqslant k \\ n_s\mid m-a_s}}\Gamma\left(\frac{x+a_s}{n_s}\right)}{\prod_{\substack{1\leqslant t\leqslant l \\ m_t\mid m-b_t}}\Gamma\left(\frac{x+b_t}{m_t}\right)}.$$

Letting $x \to -m$ we obtain from ($\star$) and ($*$) that

$$\prod_{\substack{1\leqslant s\leqslant k \\ n_s\mid m-a_s}}\left(\frac{x+m}{n_s}\right)^{-1}\Bigg/\prod_{\substack{1\leqslant t\leqslant l \\ m_t\mid m-b_t}}\left(\frac{x+m}{m_t}\right)^{-1} \longrightarrow c \text{ for some } c \in \mathbb{C}^*.$$

Thus $(x+m)^{w_B(m)-w_A(m)}$ tends to a nonzero number as $x \to -m$. This shows that $w_A(m) = w_B(m)$. We are done.

Remark 3.1. (a) When (3.1) holds with $n_1 \leqslant \cdots \leqslant n_{k-1} < n_k$ and $m_1 \leqslant \cdots \leqslant m_l$, by taking $c = 1/n_k$ in (3.2) we obtain that $1/n_k = \sum_{t\in J} 1/m_t$ for some $J \subseteq$

$\{1,\cdots,l\}$ and hence $n_k \leqslant m_l$. From this we can deduce that if $A=\{a_s(n_s)\}_{s=1}^k$ and $B=\{b_t(m_t\}_{t=1}^l$ are equivalent systems with $n_1<\cdots<n_k$ and $m_1<\cdots<m_l$ then $A=B$ (i.e. $k=l$, $n_s=m_s$ and $a_s=b_s$ for $s=1,\cdots,k$), this uniqueness result was discovered by S. K. Stein [S]. When $B=\{b_t(m_t)\}_{t=1}^l$ is the system of m copies of $0(1)$, the right hand side of the formula in (3.2) turns out to be

$$\sum_{\substack{J\subseteq\{1,\cdots,m\}\\ |J|=\sum_{t\in J}1=c}}(-1)^{|J|}=\begin{cases}(-1)^n\binom{m}{n} & \text{if } c=n \text{ for some } n=0,1,\cdots,m,\\ 0 & \text{otherwise.}\end{cases}$$

Thus (1.1) forms an exact m-cover of $\mathbb{Z}$ if and only if

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \sum_{s\in I}\frac{1}{n_s}=n}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}\frac{a_s}{n_s}}=(-1)^n\binom{m}{n}\quad\text{for } n=1,\cdots,m$$

and

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \sum_{s\in I}\frac{1}{n_s}=\sum_{s\in J}\frac{1}{n_s}}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}\frac{a_s}{n_s}}=0\quad\text{for any } J\subseteq\{1,\cdots,k\}\text{ with }\sum_{s\in J}\frac{1}{n_s}\notin\mathbb{Z}.$$

For connections of covers of $\mathbb{Z}$ with Egyptian fractions, the reader may see [Su3–Su7].

(b) By the proof of '(3.3)⇒(3.1)' and '(3.4)⇒(3.1)', (3.1) is valid if the formula in (3.3) or (3.4) holds for any $z\in\mathbb{C}\setminus\mathbb{Z}$. Thus (3.1) has the following two equivalent forms by analytic continuation.

$$(\dagger)\qquad 2^{k-l}\prod_{s=1}^k\sin\pi\frac{a_s-z}{n_s}=\prod_{t=1}^l\sin\pi\frac{b_t-z}{m_t}\ \text{for all } z\in\mathbb{C};$$

$$(\ddagger)\quad \prod_{s=1}^k\Gamma\left(\frac{z+a_s}{n_s}\right)n_s^{\frac{z+a_s}{n_s}-\frac12}=(2\pi)^{\frac{k-l}{2}}\prod_{t=1}^l\Gamma\left(\frac{z+b_t}{m_t}\right)m_t^{\frac{z+b_t}{m_t}-\frac12}\ \text{for } z\in\mathbb{C}\setminus-\mathbb{N}.$$

That (†) implies (3.1) was first obtained by Stein [S] in the case $B=\{0(1)\}$; the converse in general case was noticed by the author in 1989 as a consequence of theorems in [Su1], when $B=\{b_t(m_t)\}_{t=1}^l$ is simply $\{0(1)\}$ it was found repeatedly by J. Beebee [Be1] in 1991. That (3.1) implies (‡) is essentially Corollary 3 of Sun [Su1] which extends the Gauss multiplication formula, the converse was mentioned in [Su1] as a conjecture. By the above, (1.1) is an exact m-cover of $\mathbb{Z}$ (i.e. $A\sim\{\langle m,0,1\rangle\}$) if and only if

$$(3.6)\qquad \prod_{s=1}^k\left(\Gamma\left(\frac{z+a_s}{n_s}\right)n_s^{\frac{z+a_s}{n_s}-\frac12}\right)=(2\pi)^{\frac{k-m}{2}}\Gamma^m(z)\ \text{ for all } z\in\mathbb{C}\setminus-\mathbb{N}.$$

Consequently, if (1.1) is an exact 1-cover of $\mathbb{Z}$ with $a_1 = 0$, then

$$\frac{n_1^{\frac{z}{n_1}-\frac{1}{2}}\prod_{s=2}^k\left(\Gamma\left(\frac{z+a_s}{n_s}\right)n_s^{\frac{z+a_s}{n_s}-\frac{1}{2}}\right)}{n_1^{-\frac{1}{2}}\prod_{s=2}^k\left(\Gamma\left(\frac{a_s}{n_s}\right)n_s^{\frac{a_s}{n_s}-\frac{1}{2}}\right)}=\frac{(2\pi)^{\frac{k-1}{2}}\Gamma(z)/\Gamma\left(\frac{z}{n_1}\right)}{(2\pi)^{\frac{k-1}{2}}\lim_{z'\to 0}\frac{\Gamma(z')z'}{\Gamma\left(\frac{z'}{n_1}\right)\frac{z'}{n_1}\cdot n_1}}$$

for $z \neq 0, -1, -2, \cdots$, i.e.,

$$\Gamma(z)=\frac{\Gamma\left(\frac{z}{n_1}\right)}{n_1^{1-\frac{z}{n_1}}}\prod_{s=2}^k\frac{\Gamma\left(\frac{z+a_s}{n_s}\right)}{n_s^{-\frac{z}{n_s}}\Gamma\left(\frac{a_s}{n_s}\right)}\quad\text{for all } z\in\mathbb{C}\setminus -\mathbb{N} \tag{3.7}$$

(by Theorem 3.1 we directly have $\prod_{s=2}^k\Gamma(\frac{a_s}{n_s})n_s^{a_s/n_s-1/2} = (2\pi)^{(k-1)/2}n_1^{-1/2}$). In 1994 Beebee [Be3] showed that the relative formula (3.7) holds for any exact 1-cover (1.1) of $\mathbb{Z}$ as we have seen this was actually rooted in [Su1] published in 1989. The new contribution of [Be3] is that if (3.7) holds then (1.1) must be an exact 1-cover of $\mathbb{Z}$, however we have a simpler equivalent form (‡) of (3.1).

Now we give

Theorem 3.2. *For every $s = 1, \cdots, k$ we let $\lambda_s \in \mathbb{C}$, $n_s \in \mathbb{Z}^+$ and $a_s \in R(n_s)$. Then the following statements are equivalent:*

$$\mathcal{A} = \{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^k \sim \emptyset; \tag{3.8}$$

$$\sum_{s=1}^k\frac{(\lambda_s^{(t)})^2}{n_s}=-2\sum_{\substack{1\leqslant i<j\leqslant k\\ \gcd(n_i,n_j)\mid a_i-a_j}}\frac{\lambda_i^{(t)}\lambda_j^{(t)}}{[n_i,n_j]}\quad\textit{for } t=1,2 \tag{3.9}$$

where $\lambda_s^{(1)} = \mathrm{Re}\lambda_s$ and $\lambda_s^{(2)} = \mathrm{Im}\lambda_s$ for any $s = 1, \cdots, k$;

$$\sum_{s=1}^k\lambda_s\left(\left[\frac{x+ma_s}{n_s}\right]-\frac{m-1}{2}\right)=0\quad\textit{for all } x\in\mathbb{Z} \tag{3.10}$$

where m is an integer prime to the moduli $n_1, \cdots, n_k$;

$$\sum_{s=1}^k\lambda_s\cot_m\left(\frac{z+a_s}{n_s},n_s\right)=0\qquad\textit{for all } z\in\mathbb{C} \tag{3.11}$$

where m is a nonnegative integer and $\cot_z^{(m)}$ is as in Example 2.4(ii);

$$\sum_{t=1}^k\lambda_t\zeta_s\left(\frac{z+a_t}{n_t},n_t\right)=0\quad\textit{for all } z\in\mathbb{C}\setminus(-\infty,0] \tag{3.12}$$

where s is a complex number not in $-\mathbb{N}$ and ζ_s is as in Example 2.3(ii)*.*

Proof. (3.8) $\Leftrightarrow$ (3.9). For $t=1,2$ and $N=[n_1,\cdots,n_k]$ we can easily check that

$$\frac{1}{N}\sum_{r=0}^{N-1}\Bigg(\sum_{\substack{1\leqslant s\leqslant k\\ n_s\mid r-a_s}}\lambda_s^{(t)}\Bigg)^2=\sum_{s=1}^k\frac{(\lambda_s^{(t)})^2}{n_s}+2\sum_{\substack{1\leqslant i<j\leqslant k\\ a_i(n_i)\cap a_j(n_j)\neq\emptyset}}\frac{\lambda_i^{(t)}\lambda_j^{(t)}}{[n_i,n_j]}.$$

So (3.8) and (3.9) are equivalent.

(3.8) $\Leftrightarrow$ (3.10). Let $\mathcal{B}=\{\langle\lambda_s,b_s,n_s\rangle\}_{s=1}^k$ where b_s is the least nonnegative residue of ma_s mod n_s. As m is prime to $N=[n_1,\cdots,n_k]$, any integer z can be written in the form $mu+Nv$ (with $u,v\in\mathbb{Z}$) and hence $w_{\mathcal{B}}(z)=w_{\mathcal{B}}(mu)=w_{\mathcal{A}}(u)$. Thus $\mathcal{B}\sim\emptyset$ if $\mathcal{A}\sim\emptyset$. Clearly $f(x,y)=x-\frac{1}{2}$ and $g(x,y)=f(x,y)-[\ ](x,y)=\{x\}-\frac{1}{2}$ over $\mathbb{R}\times\mathbb{R}$ are uniform maps into $\mathbb{R}$. If $\mathcal{A}\sim\emptyset$ and $x\in\mathbb{R}$, then

$$\begin{aligned}&\sum_{s=1}^k\lambda_s\left(\left[\frac{x+ma_s}{n_s}\right]-\frac{m-1}{2}\right)\\ =&m\sum_{s=1}^k\lambda_s\left(\frac{x/m+a_s}{n_s}-\frac{1}{2}\right)-\sum_{s=1}^k\lambda_s\left(\left\{\frac{x+b_s}{n_s}\right\}-\frac{1}{2}\right)=0.\end{aligned}$$

If (3.10) holds, then so does (3.8), because for any $x\in\mathbb{Z}$ we have

$$\begin{aligned}w_{\mathcal{A}}(x)=&\sum_{\substack{1\leqslant s\leqslant k\\ n_s\mid ma_s-mx}}\lambda_s=\sum_{s=1}^k\lambda_s\left(\left[\frac{ma_s-mx}{n_s}\right]-\left[\frac{ma_s-mx-1}{n_s}\right]\right)\\ =&\sum_{s=1}^k\lambda_s\left(\left[\frac{ma_s-mx}{n_s}\right]-\frac{m-1}{2}\right)-\sum_{s=1}^k\lambda_s\left(\left[\frac{ma_s-mx-1}{n_s}\right]-\frac{m-1}{2}\right)=0.\end{aligned}$$

(3.8) $\Leftrightarrow$ (3.11). Since $\cot_m$ is a uniform map into $\mathbb{C}$, (3.8) implies (3.11) by Theorem 1.1. By Example 2.4(ii),

$$\cot_m(z,1)=\frac{m!}{\pi^{m+1}}\sum_{k=-\infty}^{+\infty}\frac{1}{(k+z)^{m+1}}\quad\text{for } z\in\mathbb{C}\setminus\mathbb{Z}.$$

Obviously $\cot_m(z,1)\to\infty$ as z tends to an integer. If $n\in\mathbb{Z}^+$, then $\cot_m(z,n)=\cot_m(z,1)/n_{m+1}$ is continuous for $z\in\mathbb{C}\setminus\mathbb{Z}$. In the light of Lemma 3.1, (3.11) implies (3.8).

(3.8) $\Leftrightarrow$ (3.12). By Example 2.3(ii), ζ_s is a uniform map into $\mathbb{C}$. So (3.12) is implied by (3.8).

As $s\neq 0,-1,-2,\cdots$, by Example 2.3(ii) we have

$$\frac{d}{dv}\zeta(s,v)=-s\zeta(s+1,v),\ \frac{d}{dv}\zeta(s+1,v)=(-s-1)\zeta(s+2,v),\ \cdots$$

in the region $\mathbb{C}\setminus(-\infty,0]$. Let $m=[2-\mathrm{Re}(s)]$ if $\mathrm{Re}(s)\leqslant 1$, and $m=0$ if $\mathrm{Re}(s)>1$. Put $\bar{s}=s+m$. Then $\mathrm{Re}(\bar{s})>1$ and

$$\frac{d^m}{dv^m}\zeta(s,v)=\prod_{0\leqslant j<m}(-s-j)\cdot\zeta(\bar{s},v)\quad\text{for } v\in\mathbb{C}\setminus(-\infty,0].$$

If $n\in\mathbb{Z}^+$ and $a\in R(n)$, then

$$\frac{d^m}{dz^m}\zeta_s\left(\frac{z+a}{n},n\right)=(-1)^m s(s+1)\cdots(\bar{s}-1)\zeta_{\bar{s}}\left(\frac{z+a}{n},n\right)\ \text{for } z\in\mathbb{C}\setminus(-\infty,0].$$

Apparently $\zeta_{\bar{s}}(z,1)=\zeta(\bar{s},z)\to\infty$ as z tends to an integer in $-\mathbb{N}$.

Let's assume (3.12). Then

$$\sum_{t=1}^k\lambda_t\zeta_{\bar{s}}\left(\frac{z+a_t}{n_t},n_t\right)=\frac{(-1)^m}{s(s+1)\cdots(\bar{s}-1)}\frac{d^m}{dz^m}\sum_{t=1}^k\lambda_t\zeta_s\left(\frac{z+a_t}{n_t},n_t\right)=0$$

for $z\in\mathbb{C}\setminus(-\infty,0]$, and hence by analytic continuation

$$\sum_{t=1}^k\lambda_t\zeta_{\bar{s}}\left(\frac{z+a_t}{n_t},n_t\right)=0\qquad\text{for all } z\neq 0,-1,-2,\cdots.$$

Applying Lemma 3.1 we then get (3.8).

So far we have completed the proof of Theorem 3.2.

Remark 3.2. In the case $m=1$, that (3.8) implies (3.10) was first realized by the author [Su1] in 1989 and later refound by Porubský [P4] in 1994. If (3.8) holds, then the formula in (3.10) in the case $m=1$ and $x=[n_1,\cdots,n_k]$ yields the equality $\sum_{s=1}^k\frac{\lambda_s}{n_s}=0$. In 1989 the author [Su1] obtained Theorem 1.1 and noted that $f(x,y)=\frac{1}{y}\cot\pi x$ over $(\mathbb{C}\setminus\mathbb{Z})\times\mathbb{C}^*$ is a uniform map into $\mathbb{C}$, thus for any exact 1-cover (1.1) we have

$$\sum_{s=1}^k\frac{1}{n_s}\cot\left(\pi\frac{z+a_s}{n_s}\right)=\cot\pi z\ \text{ and }\ \sum_{s=1}^k\frac{1}{n_s^2}\csc^2\left(\pi\frac{z+a_s}{n_s}\right)=\csc^2(\pi z)$$

for all $z\in\mathbb{C}\setminus\mathbb{Z}$, this was also given by Beebee [Be1] in 1991.

Corollary 3.1. *Let* (1.1) *be a finite system of arithmetic sequences, and m a positive integer. Then*

(i) (1.1) *forms an exact m-cover of $\mathbb{Z}$ if and only if*

$$\sum_{s=1}^k\frac{1}{n_s}=m\quad\text{and}\quad\sum_{\substack{1\leqslant i<j\leqslant k\\(n_i,n_j)\mid a_i-a_j}}\frac{1}{[n_i,n_j]}=\frac{m(m-1)}{2},\tag{3.13}$$

also (1.1) *forms an exact m-cover of $\mathbb{Z}$ if and only if*

$$\sum_{s=1}^{k}\frac{1}{n_s}=m \quad \text{and} \quad \sum_{s=1}^{k}\left[\frac{a+na_s}{n_s}\right]=am+\frac{(k-m)(n-1)}{2} \text{ for } a\in R(N) \tag{3.14}$$

where n is any fixed integer prime to $N=[n_1,\cdots,n_k]$.

(ii) Suppose that (1.1) is an m-cover of $\mathbb{Z}$. Then

$$\sum_{t=1}^{k} n_t^{-s}\zeta\left(s,\frac{x+a_t}{n_t}\right)\geqslant m\zeta(s,x) \quad \text{for } s>1 \text{ and } x>0, \tag{3.15}$$

and for any $n\in\mathbb{Z}^+$ we have

$$\begin{aligned}&\sum_{\substack{1\leqslant s\leqslant k\\ x+a_s\notin n_s\mathbb{Z}}}\frac{1}{n_s^{2n}}\cot^{(2n-1)}\left(\pi\frac{x+a_s}{n_s}\right)\\ &\leqslant\begin{cases} m\cot^{(2n-1)}(\pi x) & \text{if } x\in\mathbb{R}\setminus\mathbb{Z},\\ (-4)^n\frac{B_{2n}}{2n}(m-\sum_{\substack{1\leqslant s\leqslant k\\ n_s\mid x+a_s}}\frac{1}{n_s^{2n}}) & \text{if } x\in\mathbb{Z}.\end{cases}\end{aligned} \tag{3.16}$$

Proof. Let $\mathcal{A}=\{\langle 1,a_1,n_1\rangle,\cdots,\langle 1,a_k,n_k\rangle,\langle -m,0,1\rangle\}$.

i) Clearly $A=\{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover if and only if $\mathcal{A}\sim\emptyset$. If $\mathcal{A}\sim\emptyset$, then $\sum_{s=1}^k 1/n_s-m=0$ by Remark 3.2. (That $\sum_{s=1}^k 1/n_s=m$ for any exact m-cover (1.1) is actually a well-known result, it can be found in [P2].)

By the equivalence of (3.8) and (3.9), $\mathcal{A}\sim\emptyset$ if and only if

$$\sum_{s=1}^{k}\frac{1}{n_s}+\frac{(-m)^2}{1}=-2\Bigg(\sum_{\substack{1\leqslant i<j\leqslant k\\ (n_i,n_j)\mid a_i-a_j}}\frac{1}{[n_i,n_j]}+\sum_{\substack{1\leqslant s\leqslant k\\ (n_s,1)\mid a_s-0}}\frac{-m}{n_s}\Bigg) \tag{$\triangle$}$$

Under the condition $\sum_{s=1}^k 1/n_s=m$, ($\triangle$) reduces to the latter equality in (3.13). So, $\mathcal{A}\sim\emptyset$ if and only if (3.13) holds.

By Theorem 3.2, $\mathcal{A}\sim\emptyset$ if and only if for all $x\in\mathbb{Z}$ we have

$$\sum_{s=1}^{k}\left(\left[\frac{x+na_s}{n_s}\right]-\frac{n-1}{2}\right)-m\left(\left[\frac{x+n\cdot 0}{1}\right]-\frac{n-1}{2}\right)=0,$$

i.e.

$$\sum_{s=1}^{k}\left[\frac{x+na_s}{n_s}\right]=mx+(k-m)\frac{n-1}{2}.$$

Any integer x can be written in the form $a+qN$ where $a\in R(n)$ and $q\in\mathbb{Z}$, thus the last equality holds for all $x\in\mathbb{Z}$ if and only if (3.14) is valid. This ends the proof of part (i).

ii) As $w_A(x)\geqslant m$ for all $x\in\mathbb{Z}$, $w_{\mathcal{A}}(x)\geqslant 0$ for any $x\in\mathbb{Z}$. Obviously $\mathcal{A}\sim\{\langle w_{\mathcal{A}}(r),r,N\rangle\}_{r=0}^{N-1}$. When $s>1$ and $x>0$, by Theorem 3.2

$$\sum_{t=1}^{k}\zeta_s\left(\frac{x+a_t}{n_t},n_t\right)-m\zeta_s\left(\frac{x+0}{1},1\right)=\sum_{r=0}^{N-1}w_{\mathcal{A}}(r)\zeta_s\left(\frac{x+r}{N},N\right),$$

therefore

$$\sum_{t=1}^{k}\frac{1}{n_t^s}\zeta\left(s,\frac{x+a_t}{n_t}\right)-m\zeta(s,x)=\sum_{r=0}^{N-1}\frac{w_{\mathcal{A}}(r)}{N^s}\zeta\left(s,\frac{x+r}{N}\right)\geqslant 0$$

since $\zeta(s,\frac{x+r}{N})=\sum_{j=0}^{\infty}(j+\frac{x+r}{N})^{-s}>0$.

By Example 2.4(ii),

$$\cot_{2n-1}(x,N)=\frac{(2n-1)!}{(\pi N)^{2n}}\sum_{\substack{j=-\infty\\ j\neq -x}}^{+\infty}\frac{1}{(j+x)^{2n}}>0 \text{ for all } x\in\mathbb{R}.$$

As in the last paragraph, now we have

$$\sum_{s=1}^{k}\cot_{2n-1}\left(\frac{x+a_s}{n_s},n_s\right)-m\cot_{2n-1}(x,1)\geqslant 0\quad\text{for any } x\in\mathbb{R}.$$

Clearly this is equivalent to (3.16). We are done.

Remark 3.3. (3.16) in the case $n=1$ gives the following inequality:

$$(3.17)\qquad \sum_{\substack{1\leqslant s\leqslant k\\ x+a_s\notin n_s\mathbb{Z}}}\frac{1}{n_s^2}\csc^2\left(\pi\frac{x+a_s}{n_s}\right)\geqslant\begin{cases} m\csc^2(\pi x) & \text{if } x\in\mathbb{R}\setminus\mathbb{Z},\\ \frac{1}{3}(m-\sum_{\substack{1\leqslant s\leqslant k\\ n_s\mid x+a_s}}\frac{1}{n_s^2}) & \text{if } x\in\mathbb{Z}.\end{cases}$$

Let $\mathcal{A}=\{\langle\lambda_s,a_s,n_s\rangle\}_{s=1}^{k}\in\mathrm{S}(\mathbb{C})$ and $z\in\mathbb{C}$. For $y\in(-2\pi/\max\{n_1,\cdots,n_k\},0)$ we have

$$\begin{aligned}-e^{yz}\sum_{n=0}^{\infty}w_{\mathcal{A}}(n)(e^y)^n&=-\sum_{s=1}^{k}\lambda_s\frac{(e^y)^{z+a_s}}{1-(e^y)^{n_s}}\\ =\sum_{s=1}^{k}\frac{\lambda_s}{n_sy}\cdot\frac{n_sye^{n_sy\cdot\frac{z+a_s}{n_s}}}{e^{n_sy}-1}&=\sum_{s=1}^{k}\frac{\lambda_s}{n_sy}\sum_{n=0}^{\infty}\frac{B_n(\frac{z+a_s}{n_s})}{n!}(n_sy)^n\end{aligned}$$

where $B_n(x)$ is the Bernoulli polynomial of degree n. So, $\mathcal{A}\sim\emptyset$ if and only if

$$(3.18)\qquad \sum_{s=1}^{k}\lambda_sn_s^{n-1}B_n\left(\frac{z+a_s}{n_s}\right)=0\ \text{ for all } n=0,1,2,\cdots.$$

For the system $\mathcal{B}=\{\langle 1,a_1,n_1\rangle,\cdots,\langle 1,a_k,n_k\rangle,\langle -m,0,1\rangle\}$, this was proved by A. S. Fraenkel [F1,F2] in the case $m=1$ and $z=0$, by Beebee [Be2] in the case $m=1$, and by Porubský [P2] in the case $m\in\mathbb{Z}^+$ and $z=0$. See also Porubský [P1,P3] and Znám [Z] for the case $z=0$ with the weights 1 in $\mathcal{B}$ replaced by real weights. In 1994 Porubský [P4] essentially established the above general result. However, before the works of Beebee [Be2] and Porubský [P4], in 1989 the author [Su1] proved Theorem 1.1 and observed that the function $b_n(x,y)=y^{n-1}B_n(x)$ is a uniform map into $\mathbb{C}$ for each $n\in\mathbb{N}$. In 1988 D.H. Lehmer [Le] showed that $B_n(x)$ is the only monic polynomial of degree n such that

$$\sum_{r=0}^{d-1}d^{n-1}B_n\left(\frac{x+r}{d}\right)=B_n(x)\quad\text{for all } d\in\mathbb{Z}^+.$$

For any $n\in\mathbb{N}$ clearly

$$\sum_{s=1}^{k}\lambda_s n_s^{n-1}B_n\left(\frac{z+a_s}{n_s}\right)=\sum_{l=0}^{n}\binom{n}{l}B_l\sum_{s=1}^{k}\lambda_s n_s^{l-1}(z+a_s)^{n-l}.$$

Thus $\mathcal{A}\sim\emptyset$ if and only if

$$\sum_{l=0}^{n}\binom{n}{l}B_l\sum_{s=1}^{k}\lambda_s n_s^{l-1}a_s^{n-l}=0\quad\text{for } n=0,1,2,\cdots. \tag{3.19}$$

In 1991 E. Y. Deeba and D. M. Rodriguez [DR] found this for the trivial system $\{\langle 1,0,d\rangle,\langle 1,1,d\rangle,\cdots,\langle 1,d-1,d\rangle,\langle -1,0,1\rangle\}$ where $d\in\mathbb{Z}^+$, later Beebee [Be2] obtained the result for system $\mathcal{B}$ with $m=1$, and Porubský [P5] observed the generalization to $\mathcal{A}\in\mathrm{S}(\mathbb{R})$.

For the covering equivalence between systems in $\mathrm{S}(\mathbb{R})$, Porubský [P5] provided some characterizations involving Euler polynomials and recursions for Euler numbers.

In [Su2] the author announced several results closely related to this paper, proofs of them are presented in a recent paper [Su9].

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